TWO PROJECTS OF B. SHAPIRO

BORIS SHAPIRO

1. Waring problem for polynomials and superFermat problem

In recent years there have been a lot of activity in the area of additive decompositions of polynomials which is the problem of presentation of a given polynomial/form with coefficients in a certain field as a sum of polynomials of a certain form. One classical issue of this problem is to find a presentation of a form of degree \(dk\) in a given number of variables as a sum of \(k\)th powers of forms of degree \(d\). This problem has been addressed in e.g. [8, 10, 7]. In spite of a serious progress in this area there are still numerous problems to be addressed.

A superFermat equation has the form

\[
y_1^n + y_2^n + \cdots + y_l^n = y_{l+1}^{n+1},
\]

where \(n \geq 2\) and \(y_1, y_2, \ldots, y_{l+1}\) belong to some semi-ring \(C\), the classical case being \(C = \mathbb{N}\). Fermat’s last theorem (also called Fermat’s conjecture) settled by A. Wiles in 1995, claims that (1.1) has no solution in positive integers for \(l = 2\) and \(n \geq 3\). Already L. Euler considered the following natural problem. Given \(n \geq 2\), find the minimal number \(g(n)\) such that (1.1) has a solution in positive integers. He observed that \(g(2) = 2\) and \(g(3) = 3\) and conjectured that \(g(n) \geq n\), which was disproved by N. Elkies not that long ago.

The main emphasis of the suggested project is to look at the case of other natural semigroups \(C\). Let us consider solutions of (1.1) in homogeneous forms of a given degree \(d\) in \(n + 1\) homogeneous variables \((x_0, x_1, \ldots, x_n)\) with complex coefficients, comp. [3] and [9]. We say that a solution \((p_1, \ldots, p_{l+1})\) of (1.1) in homogeneous forms is very trivial if all forms \(p_j, j = 1, \ldots, l + 1\) are proportional to one and the same non-zero form. The set of all very trivial solutions in forms of degree \(d\) is a product of the projective space of all non-zero forms of degree \(d\) (considered up to a scalar factor) times the hypersurface of all solutions of (1.1) in complex numbers. We say that a solution \((p_1, \ldots, p_{l+1})\) of (1.1) in homogeneous forms is trivial if it can be partitioned in the union of very trivial solutions for smaller values of \(l\). The remaining solutions of (1.1) are called non-trivial.

**Problem 1.** For which triples \((n, l, d)\), there exist non-trivial solutions of (1.1) in forms of degree \(d\) in \(n + 1\) homogeneous variables?

For \(l = 2\), \(n = 1\) and any \(k \geq 2\), an answer to Problem 1 follows from a result of Liouville, see [11], p. 263.

For this project a successful applicant needs to have a reasonable knowledge of commutative algebra and/or algebraic geometry.

2. Inverse problem in Polya-Schur theory

The main question considered in the Pólya–Schur theory [6] can be formulated as follows.

**Problem 2.** Given a subset \(S \subset \mathbb{C}\) of the complex plane, describe the semigroup of all linear operators \(T : \mathbb{C}[z] \rightarrow \mathbb{C}[z]\) sending any polynomial with roots in \(S\) to a polynomial with roots in \(S\) (or to 0).
Definition 1. If an operator $T$ has the latter property, then we say that $S$ is a $T$-invariant set, or that $T$ preserves $S$.

So far Problem 2 has only been solved for the circular domains (i.e., images of the unit disk under Möbius transformations), their boundaries [4], and recently for strips [5]. Even a very similar case of the unit interval is still open at present. It seems that for a somewhat general class of subsets $S \subseteq \mathbb{C}$, is out of reach of all currently existing methods.

In this project, we consider an inverse problem in the Pólya–Schur theory which seems both natural and more accessible. We will restrict ourselves to consideration of closed $T$-invariant subsets.

Problem 3. Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, find a sufficiently large class of closed $T$-invariant sets. Ultimately, describe all closed $T$-invariant subsets of the complex plane.

A large number of initial results were obtained in [1, 2], but there is still an abundance of intriguing related questions still waiting to be addressed.

For this project a successful applicant needs to have a good background in analysis and complex dynamics.

References