Proposal for a Constructive Tarski-Grothendieck Set Theory

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It is known since the work of Aczel (1986) that a good constructive counterpart to *regular cardinal* is the notion of regular set. In CZF a *regular set* is a set A such that A is inhabited and transitive, and that moreover, for every set $a \in A$, and every set $R \subseteq a \times A$, with the property $(\forall x \in a)(\exists y \in A)\langle x, y \rangle \in R$, there exists a set $c \in A$ such that $(\forall x \in a)(\exists y \in c)\langle x, y \rangle \in R$, and $(\forall y \in c)(\exists x \in a)\langle x, y \rangle \in R$. In particular, if R is a function $a \longrightarrow A$, then the image of a under R is included in a set $c \in A$.

In Rathjen *et al.* (1998) a constructive theory of inaccessible sets is presented and interpreted in constructive set theory (following Aczel 1986). In CZF a set I is *set-inaccessible* if I is regular, such that $(\forall x \in I)(\exists y \in I)(x \subseteq y \wedge \; y \text{ is regular})$, and moreover that $(I, \in_{|(I \times I)})$ is a first-order model of CZF.

An obvious suggestion for a *Tarski-Grothendieck axiom* for CZF is then

$$(\forall x)(\exists I)(x \in I \wedge \; I \text{ is set-inaccessible}).$$

References

Peter Aczel. The type theoretic interpretation of constructive set theory: inductive definitions, in: R.B. Marcus et al. (Eds.), *Logic, Methodology. and Philosophy of Science VII,* North-Holland, Amsterdam, 1986.

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Tarski-Grothendieck Set Theory. Wikipedia https://en.wikipedia.org/wiki/Tarski-Grothendieck_set_theory (retrieved February 28, 2019)