$\Lambda(\Theta)$-MODULES AND ALGEBRAIC VECTOR BUNDLES ON $\mathbb{P}(\Theta)$

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1. Introduction

1.1. The main result of this paper is the reduction of the problem on classification of algebraic vector bundles on n-dimensional projective space $\mathbb{P}^n$ to a problem of Linear Algebra, namely to that of construction of finite dimensional graded modules over the Grassmann algebra. We mean the following.

Let the n-dimensional projective space $\mathbb{P}^n$ be the projectivization (the space of lines) of the $(n+1)$-dimensional linear space Θ.

In Θ, choose a basis $\xi_0, \xi_1, \dots, \xi_n$. In the Grassmann algebra $\Lambda := \Lambda(\Theta)$ introduce the $\mathbb{Z}$-grading by setting $\deg \xi_i = 1$ for any i.

A $\mathbb{Z}$-graded Λ-module is a vector space V endowed with a decomposition $V = \oplus V_j$, so that the $\Lambda(\Theta)$-action is consistent with the grading. This means that for any i, where $0 \leq i \leq n$, the linear mapping $\tilde{\xi}_i : V \longrightarrow V$ is defined, so that

1) $\tilde{\xi}_i V_j \subset V_{j+1}$;
2) $\tilde{\xi}_i \tilde{\xi}_j + \tilde{\xi}_j \tilde{\xi}_i = 0$ as linear mappings of V into V.

In the class of all $\Lambda(\Theta)$-modules the subclass of so-called *nice* modules is distinguished (for the definition see in sec. 1.6). From any nice $\Lambda(\Theta)$-module V the algebraic vector bundle $\Phi(V)$ on $\mathbb{P}^n$ is recovered (see sec. 2.5). The main result (Theorem 2.6) claims that any algebraic vector bundle on $\mathbb{P}^n$ is isomorphic to the bundle $\Phi(V)$ for a nice $\Lambda(\Theta)$-module V; it also describes, when two such bundles $\Phi(V)$ and $\Phi(V')$ are isomorphic.

1.2. Due to the classical Serre theorem (see, e.g., [Ha2]) algebraic vector bundles (and more general objects, i.e., coherent sheaves) on $\mathbb{P}^n$ are described by a finitely generated graded modules over the algebra $S(X)$ of polynomials in $n+1$ commuting indeterminates $x_0, \dots, x_n$. Our description of bundles in terms of modules over the Grassmann algebra $\Lambda(\Theta)$ is based on the profound duality between $S(X)$ and $\Lambda(\Theta)$. Namely, let $\mathcal{M}^b(S)$ and $\mathcal{M}^b(\Lambda)$ be categories of finitely generated graded modules over $S(X)$ and $\Lambda(\Theta)$, respectively.

Let us consider the corresponding derived categories $D^b(S)$ and $D^b(\Lambda)$. (Roughly speaking the derived category is the quotient of the category of the complexes of corresponding modules by the category of complexes with zero cohomology; for details, see section 5, Appendix and [Ha1], [V].) It turns out (Theorem 5.7) that categories $D^b(S)$ and $D^b(\Lambda)$ are equivalent. The proof of Theorem 3.7 is based on the Koszul complex and is, surprisingly, very simple.

1.3. The study of algebraic vector bundles on projective spaces attracts now a good deal of interest. The Heartshorn's problem list [Ha3] and the review [S] are good introduction to the achievement and problems in the field.

On the one hand, there is a number of problems arising in the algebraic geometry itself (see, e.g., [S]). On the other hand, as was recently discovered, these bundles are also directly connected with several solutions of important equations of the classical field theory: with the so-called mani-instanton solutions of the Young-Mills equation (see [DM], [Ha3]).

The results of the present paper which were earlier announced in the notice of the authors and I.M. Gelfand [BGG] , appeared, when we thought over the talk of Yu.I. Manin on his joint paper with V.G. Drinfeld [DM]. Another approach to the classification of algebraic vector bundles was proposed by A.A. Beilinson [B]. He reduced this classification to the linear algebra problem different from the one we considered.

1.4. Contents. The first two sections provide with background necessary for the exact formulation of the main theorem. They do not require almost any preliminary knowledge. In the following four sections we prove the theorem. The main tool — that of derived categories — is not yet a common knowledge. Therefore, in §§3–6, we give sufficiently detailed proofs reproducing parts of papers by Heartshorn [Ha1] and

Verdier [V]. Several notions and results on derived categories needed are given in Appendix. Besides, in §6 there is shown the relation between the construction of vector bundles due to BGG and that by Beilinson [B].

2. The modules over the Grassmann algebra

2.1. In the sequel an algebraically closed field $\mathbb{K}$ is fixed; all linear spaces, polynomials, and so on are considered over this field unless otherwise mentioned. Without detriment to understanding the reader may assume that $\mathbb{K} = \mathbb{C}$.

2.2. In the sequel the space Θ will be fixed so instead of $\Lambda(\Theta)$ we will write just Λ.

It is clear that Θ is embedded in $\Lambda(\Theta)$. It is easy to see that $\theta^2 = 0$ for any $\theta \in \Theta$. Conversely, the defining relations $\xi_i\xi_j + \xi_j\xi_i = 0$ follow from

$$(1) \qquad \xi_i^2 = 0, \quad \xi_j^2 = 0, \quad (\xi_i + \xi_j)^2 = 0.$$

Therefore Λ does not depend on the choice of a basis in Θ.

Denote by Λ_j the homogeneous component of degree j in Λ. Clearly, Λ_0 is one-dimensional subspace generated by the identity 1 and Λ_{n+1} is the one-dimensional subspace generated by the element $\omega = \xi_0 \dots \xi_n$.

2.3. Unless otherwise mentioned, we will say "Λ-module" for any left unitary graded module over Λ.

A *morphism* of graded Λ-modules is a linear map $\varphi\colon V \longrightarrow V'$ commuting with the Λ-action (i.e., $\varphi(\lambda v) = \lambda\varphi(v)$) which preserves the grading. The category of all Λ-modules will be denoted by $\mathcal{M}(\Lambda)$.

It is clear that the Λ-module V has a finite number of generators over Λ if and only if $\dim V < \infty$. Denote by $\mathcal{M}^b(\Lambda)$ the full subcategory of $\mathcal{M}(\Lambda)$ consisting of finitely generated Λ-modules.

2.4. Operations over Λ-modules.

a) Let $V = \oplus V_j$ be a Λ-module and i an integer. Denote by $V[i]$ the Λ-module for which $V[i]_j = V_{j-i}$ with the same Λ-action. The operation $T_i : V \longrightarrow V[i]$ will be called the *shift of grading* by i (Note that i may be any integer). Clearly, $V[i][s] = V[i+s]$

Let 1 be the trivial one-dimensional Λ-module (i.e., $\dim 1_0 = 1$, the other components being zero). Then $1[r]$ is a one-dimensional Λ-module with the only non-zero component of the degree i.

Consider Λ as the free left Λ-module with one generator of degree 0. Then $\Lambda[i]$ is the free Λ-module with one generator in degree i. An arbitrary free Λ-module is the direct sum of a certain number of modules of the form $\Lambda[i]$ for various i's.

b) The direct sum and tensor product of graded Λ-modules. Let V, $V' \in \mathcal{M}(\Lambda)$. Set

$$(2) \qquad (V \oplus V')_j = V_j \oplus V'_j; \quad (V \oplus V')_j = \oplus_\alpha V_\alpha \otimes V_{j-\alpha}$$

with the natural Λ-action:

$$(3) \qquad \xi(a \oplus b) = \xi a \oplus \xi b; \quad \xi(a \otimes b) = \xi a \otimes b + (-1)^{(\deg a)(deg \xi)} a \otimes \xi b$$

for any $a \in V$, $b \in V'$, $\xi \in \Lambda$.

The tensor product of any finite number of modules is similarly defined. Note that there are natural isomorphisms

$$(4) \qquad V \otimes V'[i] \simeq V[i] \otimes V' \simeq (V \otimes V')[i]$$

Further, for any Λ-module V the module $V \otimes \Lambda$ is free:

$$(5) \qquad V \otimes \Lambda = \oplus \Lambda[i_l]$$

and the number of modules of the form $\Lambda[i]$ in this decomposition is equal to $\dim V_i$.

c) Define the *exterior* and *symmetric powers* of the Λ-module V. First, assume that $\operatorname{char} \mathbb{K} \neq 2$. On $V^{\otimes k}$. define the action of the permutation group $\mathfrak{S}_k$ on r elements as follows:

$$(6) \qquad \sigma_i(v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_k) = (-1)^{\alpha\beta}(v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_r)$$

for $\alpha = \deg v_i$, $\beta = \deg v_{i+1}$ and σ_i, the transposition of i and $i+1$, where $1 \leq i \leq k-1$. On pairs of non-adjacent factors the formula of the $\mathfrak{S}_k$-action is more complicated.

The k-th *symmetric power* S^kV of V is the submodule in $V^{\otimes k}$ consisting of $\mathfrak{S}_k$-invariants. Similarly, the k-th exterior power Λ^kV is the submodule consisting of anti-invariant elements with respect to this $\mathfrak{S}_k$-action.

In general case (when the characteristic of $\mathbb{K}$ may be equal to 2) we must define Λ^kV and S^kV otherwise.

Let $T^{\boldsymbol{\cdot}}V = \oplus V^{\otimes k}$ be the tensor algebra of V and J_S the ideal in $T^{\boldsymbol{\cdot}}V$ generated by

$$(7) \qquad v_1 \otimes v_2 - (-1)^{\alpha\beta} v_2 \otimes v_1, \text{ where } \alpha = \deg v_1,\ \beta = \deg v_2$$

for homogeneous elements $v_1, v_2 \in V$ of odd degree. Set

$$S^{\boldsymbol{\cdot}} V = T^{\boldsymbol{\cdot}} V / J_S. \tag{8}$$

The algebra $S^{\boldsymbol{\cdot}}V$ inherits the grading of $T^{\boldsymbol{\cdot}}V$. Define S^kV as the component of degree k in $S^{\boldsymbol{\cdot}}V$. On S^kV, the Λ-action is naturally defined (starting from the Λ-action on $T^kV = \oplus V^{\otimes k}$).

Similarly, let J_A be the ideal in $T^{\boldsymbol{\cdot}}V$ generated by

$$v_1 \otimes v_2 + (-1)^{\alpha\beta} v_2 \otimes v_1, \text{ where } \alpha = \deg v_1, \ \beta = \deg v_2 \tag{9}$$

for homogeneous elements $v_1, v_2 \in V$ of even degree. In particular, $v_1 = v_2$ is allowed. Define $\Lambda^{\boldsymbol{\cdot}}V = T^{\boldsymbol{\cdot}}V/J_A$; let Λ^kV be the component of degree k in $\Lambda^{\boldsymbol{\cdot}}V$.

It is easy to see that if $\operatorname{char} \mathbb{K} \neq 2$ both definitions of S^kV and Λ^kV yield isomorphic Λ-modules.

Note that, generally speaking, $\Lambda^kV \neq 0$ for all k even for a finite dimensional Λ-module V. For example, if $V = 1[1]$, then $S^kV = 0$ for $k > 1$ whereas $\Lambda^kV = 1[k]$.

d) Let V be a left graded Λ-module. On V, we define a right Λ-module structure, e.g., by the formula

$$v\lambda = (-1)^{\alpha\beta}\lambda v, \text{ where } \alpha = \deg v, \beta = \deg \lambda. \tag{10}$$

It is clear that the ma $V \longrightarrow \bar{V}$ defines an equivalence of categories of left and right Λ-modules.

e) Finally, define the Λ-module V^*, the dual to the Λ-module V, by setting $V^* = \operatorname{Hom}_{\mathbb{K}}(\bar{V}, \mathbb{K})$.The space V^* is endowed with the left Λ-module structure by the formula

$$(\lambda\varphi)(v) = (-1)^{\beta(\alpha+\gamma)}\varphi(v\lambda), \text{ where } \alpha = \deg v, \ \beta = \deg \lambda, \ \gamma = \deg \varphi. \tag{11}$$

2.5. Complexes.

Let V be a finite dimensional $\mathbb{Z}$-graded Λ-module and $\xi \in \Theta$. Since $\deg \xi = 1$, the sequence of linear spaces and mappings given by the ξ-action

$$\dots \longrightarrow V_{i-1} \longrightarrow V_i \longrightarrow V_{i+1} \longrightarrow \dots \tag{12}$$

is a complex, i.e., the composition of any two successive mappings vanishes. Denote this complex of linear spaces by $L_\xi(V)$. Let

$$H^j(L_\xi(V)) = \operatorname{Ker}(V_j \longrightarrow V_{j+1})/\operatorname{Im}(V_{j-1} \longrightarrow V_j) \tag{13}$$

be the j-th cohomology of the complex ${}_\xi(V)$.

2.6. Nice modules.

A finitely generated Λ-module V is nice if $H^j(L_\xi(V)) = 0$ for $j \neq 0$ and any $\xi \neq 0$.

Lemma. *If V is a nice Λ-module, then $\dim H^0(L_\xi(V))$ does not depend on ξ for $\xi \neq 0$.*

Proof follows immediately from the Euler formula

$$\sum (-1)^j \dim V_j = \sum (-1)^j \dim H^j(L_\xi(V)) \tag{14}$$

valid for any finite dimensional Λ-module V.

2.7. Examples of nice Λ-modules.

(See also subsect. 2.7).

(i) The algebra Λ considered as a left Λ-module is nice. Moreover, $H^j(L_\xi(V)) = 0$ for all j and any $\xi \neq 0$. Hence, all modules $\Lambda[i]$ are nice.

(ii) The one-dimensional trivial Λ-module 1 is nice.

(iii) Let $\tilde{V}$ be the submodule in Λ generated by $\xi_0, \xi_1, \dots, \xi_n$. Then $\tilde{V}[-1]$ is nice.

(iv) If the element $\omega = \xi_0 \dots \xi_n$ generates the one-dimensional submodule (ω) in Λ. The module $\Lambda/(\omega)[-n]$ is nice.

(v) If V and V' are nice Λ-modules, then so are $V \oplus V'$, $V \otimes V'$, V^*, $S^k(V)$ and $\Lambda^k(V)$.

(vi) If in the exact sequence $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ the modules V' and V'' are nice, then so is V.

2.8. Selected properties of the category $\mathcal{M}^b(\Lambda)$.

Recall that $\mathcal{M}^b(\Lambda)$ is the category of finite dimensional (= finitely generated) graded Λ-modules. These properties are not needed to formulate Main theorem but we will need them in §§4 and 6.

Denote by $\mathcal{P}$ the subclass of free modules in $\mathcal{M}^b(\Lambda)$.

Proposition. *Let $V \in \mathcal{M}^b(\Lambda)$. The following properties are equivalent:*

(i) *V is free Λ-module;*

(ii) *V is projective Λ-module;*

(iii) *V is injective Λ- module.*

Proof. Clearly, (i) implies (ii). To prove that (ii) implies (i), note first of all that the free module Λ with one generator is indecomposable (if $\Lambda = V \oplus V'$ and, say $\dim V_0 = 1$, then $V = \Lambda$ and $V' = \{0\}$). Hence, the module $\Lambda[k]$ is indecomposable for any k. Now let V be a projective Λ-module. We may assume that V is indecomposable. Let us represent V as a direct summand of the free module F. Recall that any free module F can be represented in the form $F = \oplus\Lambda[k_i]$, where each $\Lambda[k_i]$ is indecomposable. By the Krull-Schmidt theorem, the set of indecomposable summands of V is uniquely defined. Hence V is isomorphic to one of the modules $\Lambda[k_i]$. Therefore (ii) implies (i).

To prove the equivalence of (ii) and (iii), we need one auxiliary statement. Fix a non-zero element ω in the (one-dimensional) space $\Lambda_{n+1} \subset \Lambda$.

Lemma. *For any non-zero $\lambda \in \Lambda$, there exists $\lambda' \in \Lambda$ such that $\lambda\lambda' = \omega$.*

Indeed, this λ' is the Fourier image of λ, namely

$$\lambda'(\eta) = \int e^{\eta\circ\xi}\lambda(\xi)\,\mathrm{vol}(\xi), \text{ where } \eta\circ\xi = \sum \eta_i\xi_i. \tag{15}$$

Lemma is proved.

For any k, define the pairing

$$\Lambda[k] \times \Lambda[-1-n-k] \longrightarrow \mathbb{K}, \quad (\lambda e_k, \lambda' e_[-1-k-k]) \mapsto \int(\lambda\lambda'), \tag{16}$$

where e_k and e_{-1-n-k} are generators of $\Lambda[k]$ and $\Lambda[-1-n-k]$, respectively, and $\lambda, \lambda' \in \Lambda$. Lemma implies that the pairing is non-degenerate. Therefore

$$(\Lambda[k])^* \simeq \Lambda[-k-1-n]. \tag{17}$$

Since $* : \mathcal{M}^b(\Lambda) \longrightarrow \mathcal{M}^b(\Lambda)$ is a contravariant functor and $*^2 = \mathrm{Id}$, it follows that $*$ interchanges projective and injective objects of $\mathcal{M}^b(\Lambda)$. Hence (16) and the equivalence of (i) and (ii) implies the equivalence of (i) and (iii). □

Corollary. *Any module $V \in \mathcal{M}^b(\Lambda)$ can be realized as a submodule of a module of a module from $\mathcal{P}$ and as a quotient module of a module from $\mathcal{P}$.*

Indeed, any V is a submodule of $V \otimes \Lambda[-n-1]$ and the quotient module of $V \otimes \Lambda$.

2.9. A module $V \in \mathcal{M}^b(\Lambda)$ is said to be *reduced* if $\omega V = 0$.

Lemma. (i) *An indecomposable Λ-module V is either reduced or isomorphic to ??$[k]$ for some k.*

(ii) *Any Λ-module V can be represented in the form $V = V^{fr} \oplus V^{red}$, where V^{fr} is free and V^{red} is reduced. The modules V^{fr} and V^{red} are defined uniquely up to an isomorphism.*

Proof. (i) Suppose V is not reduced, i.e., there is an element $v \in V$ such that $\omega v \neq 0$. We may assume that v is homogeneous, e.g., $v \in V_k$. Define a morphism $\varphi\colon \Lambda[k] \longrightarrow V$ by setting $\varphi(e_k) = v$, where e_k is the generator of $\Lambda[k]$. We have $\varphi(\omega e_k) = \omega\varphi(e_k) \neq 0$. Lemma ?? implies that $\varphi(\lambda e_k) \neq 0$ for any non-zero $\lambda \in \Lambda$, i.e., φ is an embedding. Since $\Lambda[k]$ is projective and V is indecomposable, we have $V \simeq \Lambda[k]$.

(ii) Let $V = V^{(1)} \oplus \cdots \oplus V^{(l)}$ be the decomposition of V into indecomposable Λ-modules. Suppose $V^{(1)}, \ldots, V^{(i)}$ are all the free summands of this decomposition. Due to heading (i) we see that $V^{fr} = V^{(1)} \oplus \cdots \oplus V^{(i)}$ and $V^{red} = V^{(i+1)} \oplus \cdots \oplus V^{(l)}$. The uniqueness of V^{fr} and V^{red} follows from the Krull-Schmidt theorem. □

2.10. Define the quotient category $\mathcal{M}^0(\Lambda) = \mathcal{M}^b(\Lambda)/\mathcal{P}$, cf. [AL]. The objects of $\mathcal{M}^0(\Lambda)$ are the same as those of $\mathcal{M}^b(\Lambda)$, but we say that the Λ-module morphism $\alpha : V \longrightarrow V'$ is *$\mathcal{P}$-equivalent to zero* if there exist a module $F \in \mathcal{P}$ and morphisms $\gamma : V \longrightarrow F$ and $\beta : F \longrightarrow V'$ such that $\alpha = \beta \circ \gamma$.

Let $L(V, V') \subset \mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, V')$ be the set of all morphisms $\mathcal{P}$-equivalent to zero. Clearly, $L(V, V')$ is a subspace in $\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, V')$. It is also clear that if $\alpha \in L(V, V')$, $\beta : V' \longrightarrow V''$ and $\gamma : V''' \longrightarrow V$, then

$$\alpha \circ \gamma \in L(V''', V'), \qquad \beta \circ \alpha \in L(V, V'').\gamma \tag{18}$$

Now, define the morphisms in $\mathcal{M}^0(\Lambda)$ by the formula

$$\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(V, V') = \mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, V')/L(V, V'). \tag{19}$$

From (1.3) it is clear that the composition or morphisms in $\mathcal{M}^b(\Lambda)$ defines a composition or morphisms in $\mathcal{M}^0(\Lambda)$. Thus, the category $\mathcal{M}^0(\Lambda)$ is completely described.

Denote by $\tau : \mathcal{M}^b(\Lambda) \longrightarrow \mathcal{M}^0(\Lambda)$ the natural functor (it is the identity on the objects and the projection on the morphisms).

2.11. Theorem. (i) *Let V and V' be reduced Λ-modules, $\alpha : V \longrightarrow V'$ a morphism. Then α is an isomorphism if and only if so is $\tau(\alpha)$. In particular, any reduced module V is recovered from $\tau(V)$ up to an isomorphism.*

(ii) *The following statements are equivalent:*

a) *$\tau(V) \simeq \tau(V')$;*

b) *There are modules $F, F' \in \mathcal{P}$ such that $V \oplus F \simeq V' \oplus F'$.*

c) *$V_{min} \simeq V'_{min}$, where V_{min} is the minimal reduced direct summand of V (see Lemma 1.11 (i)).*

(iii) *To each idempotent $p \in \mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(V, V)$ the decomposition $V = V' \oplus V''$ corresponds so that p is the projection onto V'.*

To prove the theorem we will use the following

Lemma. *$L(V, V)$ is the nilpotent ideal in $\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, V)$ for any reduced Λ-module V.*

Proof. The space $\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, V)$ is a finite dimensional $\mathbb{K}$-algebra and $L(V, V)$ is its two-sided ideal. To prove lemma, it suffices to show that in $L(V, V)$ there are no non-zero idempotents. Any idempotent $p \in \mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, V)$ corresponds to the projection of V onto (also reduced) direct summand $V' \subset V$. Let $p \in L(V, V)$ and substitute V' for V we see that $\mathrm{Id}_{V'} \in L(V', V')$, i.e., the identity map $V' \longrightarrow V'$ factors through a free module F. This means that V' is the direct summand of F; hence, contradiction in view of Lemma 1.11 (ii) since V' is reduced. □

Proof of Theorem. Heading (i) is the direct corollary of Lemma, (ii) follows from (i). To prove (iii), it suffices to note that each idempotent in the quotient of the algebra $\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, V)$ modulo the nilpotent ideal $L(V, V)$ can be lifted to an idempotent in $\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, V)$ (see ??). □

3. Sheaves on $\mathbb{P}$ and Λ-modules. Formulation of Main Theorem

3.1. Let $\mathbb{P}(\Theta)$ (or just $\mathbb{P}$) be the projective space corresponding to the linear space Θ. The points of $\mathbb{P}$ are the lines in Θ passing through the origin.

Roughly speaking, an *algebraic vector bundle* on $\mathbb{P}$ — the object of our study — is a family of vector spaces $L_{\bar{\xi}}$ algebraically depending on the point $\bar{\xi} \in \mathbb{P}$. For a precise definitions and general facts on algebraic vector bundles and also on more general objects, i.e., coherent sheaves of modules, see, e.g., the original paper by Serre [S] and a number of other books and papers, e.g., [Ha2].

We will, as a rule, identify the algebraic vector bundle $\underline{\mathrm{L}}$ on $\mathbb{P}$ with the sheaf of germs of regular sections of $\underline{\mathrm{L}}$. A coherent sheaf of modules $\mathcal{E}$ on $\mathbb{P}$ is derived from the vector bundle if and only if the latter is locally free; then the corresponding bundle is determined by $\mathcal{E}$ uniquely up to an isomorphism (see [Ha2], sec. 11.5). Hence, algebraic vector bundles are more or less the same as locally free sheaves of modules.

3.2. The most important vector bundles over $\mathbb{P}$ are the one-dimensional bundle $\mathcal{O}(1)$ and its tensor powers $\mathcal{O}(j)$. Recall the geometric description of $\mathcal{O}(1)$. Let $\bar{\xi} \in \mathbb{P}$, i.e, $\bar{\xi}$ is a line in Θ. The fiber of $\mathcal{O}(1)$ at $\bar{\xi}$ consists of all linear functions on $\bar{\xi}$.

When $\mathbb{K}$ is algebraically closed, the algebraic bundles $\mathcal{O}(j)$ are characterized as follows. Let $\bar{U}$ be an open set in $\mathbb{P}$ and $U \subset \Theta \setminus \{0\}$ the cone over $\bar{U}$, i.e., the set of points $\xi \in \Theta \setminus \{0\}$ belonging to all the lines $\bar{\xi} \in \bar{U}$. The group of sections $\Gamma(\bar{U}, \mathcal{O}(j))$ is the space of homogeneous rational functions of degree j on Θ without singularities in U.

In particular, the group $\Gamma(\mathbb{P}, \mathcal{O}(1))$ is isomorphic to the space $X = \Theta^*$, the dual of Θ. For any $x \in X$, each open set $\bar{U} \in \mathbb{P}$ and any j, the multiplication by x defines a homomorphism

$$x : \Gamma(\bar{U}, \mathcal{O}(j)) \longrightarrow \Gamma(\bar{U}, \mathcal{O}(j+1)) \tag{20}$$

i.e., a bundle morphism $x : \mathcal{O}(j) \longrightarrow \mathcal{O}(j+1)$.

3.3. Let $\mathcal{E}$ be a coherent sheaf over $\mathbb{P}$. For any point $\xi \in \mathbb{P}$, denote by $\mathcal{E}_{\bar{\xi}}$ the *geometric fiber* of the sheaf $\mathcal{E}$ at the point $\bar{\xi}$ (for definition, see [GH], [Se1], ??). Clearly, $\dim_{\mathbb{K}} \mathcal{E}_{\bar{\xi}} < \infty$. We will need the following simple description of locally free sheaves on $\mathbb{P}$.

Lemma. *The sheaf $\mathcal{E}$ is locally free if and only if $\dim \mathcal{E}_{\bar{\xi}}$ does not depend on $\xi \in \mathbb{P}$.*

For proof, see, e.g., [Se2]. □

3.4. Main Theorem. Let $V = \oplus V_j \in \mathcal{M}^b(\Lambda)$ be a finite dimensional graded Λ-module. For any j, consider the bundle

$$\mathcal{L}_j = \mathcal{L}_j(V) = V_j \otimes \mathcal{O}(j). \tag{21}$$

Here V_j is the trivial bundle on $\mathbb{P}$ with fiber V_j.

Let $\xi_0, \xi_1, \dots, \xi_n$ be a basis in Θ and $x_0, \dots, x_n$ the dual basis in X. Since V is the graded Λ-module, each element ξ_i defines the map $V_j \longrightarrow V_{j+1}$. Each element x_i defines the bundle morphism $\mathcal{O}(j) \longrightarrow \mathcal{O}(j+1)$ described in sec. 2.2. Set

$$\partial_j = \sum \xi_i \otimes x_i : \mathcal{L}_j(V) \longrightarrow \mathcal{L}_{j+1}(V). \tag{22}$$

It is easy to see that ∂_j does not depend on the choice of the basis in Θ.

Lemma. $\partial_{j+1} \circ \partial_j = 0$.

Proof immediately follows from the supercommutation relations between the ξ's and the x's. □

Thus, to any finite dimensional $\mathbb{Z}$-graded Λ-module V we have assigned the complex of vector bundles

$$\mathcal{L}(V) = \{\dots \longrightarrow \mathcal{L}_j(V) \longrightarrow \mathcal{L}_{j+1}(V) \longrightarrow \dots\}. \tag{23}$$

3.5. In $\mathcal{L}_0(V)$, consider the two sheaves $B = \operatorname{Im} \partial_{-1}$ and $Z = \ker \partial_0$. Due to Lemma 3.1?? below, $B \subset Z$, so we may consider the sheaf $\Phi(V) = Z/B$.

Lemma. *Let V be a nice Λ-module. Then Z, B and $\Phi(V)$ are locally free sheaves, i.e., the sheaves of germs of sections of some bundles.*

Proof immediately follows from Lemma 2.3, since it is easy to verify that the geometric fiber of the complex $\mathcal{L}(V)$ at point $\bar{\xi} \in \mathbb{P}$ coincides with the complex $L_{\bar{\xi}}(V)$, see sec. 1.5.

Let F be a free Λ-module. Then it is clear that $\Phi(F) = 0$. Hence, for any Λ-module V we have $\Phi(V \oplus F) = \Phi(V)$. □

Theorem. (Main Theorem) (i) *Any vector bundle $\mathcal{L}$ on $\mathbb{P}$ is of the form $\Phi(V)$ for some nice finite dimensional Λ-module V.*

(ii) $\Phi(V_1) \simeq \Phi(V_2)$ *if and only if there are free modules F_1 and F_2 such that $V_1 \oplus F_1 \simeq V_2 \oplus F_2$.*

Proof of Main Theorem will be given in §§3–5. In the remainder of this section we formulate several properties of the correspondence $V \longleftrightarrow \phi(V)$.

a) In the process of the proof we will actually prove that Φ is a functor from the complete subcategory $Pr/\mathcal{P}$ consisting of nice modules from $\mathcal{M}^b(\Lambda)/\mathcal{P}$ to the category $\operatorname{Vect}(\mathbb{P})$ of algebraic bundles on $\mathbb{P}$; the functor Φ realizes an equivalence of $Pr/\mathcal{P}$ with $\operatorname{Vect}(\mathbb{P})$.

b) The functor Φ commutes with direct sums and tensoring in both $\mathcal{M}^b(\Lambda)$ and $\operatorname{Vect}(\mathbb{P})$, with dualizations, symmetric and exterior powers.

c) The bundles corresponding to the nice modules of Example 1.7??.

(i) $\Phi(\Lambda) = 0$.

(ii) $\Phi(1) = \mathcal{O}$, the trivial one-dimensional bundle on $\mathbb{P}$.

(iii) If $V = \Lambda[-1](\xi_0, \dots, \xi_n)$, i.e., the submodule in $\Lambda[-1]$ generated by all the ξ's, then $\Phi(V) = \mathcal{O}(-1)$.

(iv) If $V = (\Lambda/(\omega))[-n]$, where $\omega = \xi_0 \dots \xi_n$, then $\Phi(V) = \mathcal{O}(1)$.

(v) Let $0 \le k \le n$. In $\Lambda[-n-1+k]$, consider the submodule L_k generated by all the elements of positive degree. Then L_k is nice and $\Phi(L_k) = \Omega^k[k]$, where Ω^k is the bundle of differential k-forms on $\mathbb{P}$.

(vi) Similarly, let T_k be the submodule in $\Lambda[-k]$ generated by all the elements of positive degree and $Q_k = \Lambda[-k]/T_k$. Then Q_k is nice and

$$\Phi(Q_k) = \Omega^{n-k}[n-k+1]. \tag{24}$$

(We will need the module Q_k in §4.)

(vii) (a generalization of (iv)). Let V be a nice module and F a free one and let

$$0 \longrightarrow V \longrightarrow F \longrightarrow \tilde{V} \longrightarrow 0 \tag{25}$$

be an exact sequence of Λ-modules. Then $\tilde{V}[-1]$ is a nice module and $\Phi(\tilde{V}[-1]) = \Phi(V) \otimes \mathcal{O}(1)$.

4. Main theorem: a scheme of the proof

4.1. In the study of algebraic bundles on the projective space $\mathbb{P}$, as well as on any other algebraic manifold, it is convenient to pass to a more general category Sh consisting of coherent sheaves of modules on $\mathbb{P}$. Assigning to any bundle the sheaf of germs of its sections we realize the category of algebraic vector bundles as the complete subcategory of Sh consisting of all locally free sheaves of modules.

4.2. In the sequel we will freely use notations and results on derived categories. Main facts on derived categories are contained in [Ha1] or [V]; for convenience's sake we formulated a number of results in Appendix.

Thus, let Sh be the category of coherent sheaves of modules on the projective space P. This category is abelian. Let $\mathcal{D}^b(\mathsf{Sh})$ be the (bounded) derived category of Sh, see sec. A.7. Recall that the objects of the category $\mathcal{D}^b(\mathsf{Sh})$ are finite complexes

$$\dots \longrightarrow \mathcal{L}_{-1} \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{L}_1 \longrightarrow \dots \tag{26}$$

of coherent sheaves on $\mathbb{P}$ such that each complex contains only finite number of non-zero terms.

4.3. Let us construct a functor $\Phi_{\mathcal{D}} : \mathcal{M}^b(\Lambda) \longrightarrow \mathcal{D}^b(\mathsf{Sh})$, generalizing the functor Φ of the previous section. We will eventually prove that $\Phi_{\mathcal{D}}$ is an equivalence of categories. That will have proved Main Theorem 2.6 and the a number of other useful results.

Let V be a finite-dimensional Λ-module and $V = \oplus V_j$ its decomposition into homogeneous components. In sec. 2.4 we have constructed the complex of bundles on $\mathbb{P}$, namely, the complex

$$\mathcal{L} = \{\dots \longrightarrow \mathcal{L}_{-1}(V) \xrightarrow{\partial_{-1}} \mathcal{L}_0(V) \xrightarrow{\partial_0} \mathcal{L}_1(V) \longrightarrow \dots\} \tag{27}$$

In this complex, $\mathcal{L}_j(V) = V_j \otimes \mathcal{O}(j)$. Since only a finite number of V_j's is non-zero, $\mathcal{L}(V)$ is a finite complex of bundles on $\mathbb{P}$.

4.4. Lemma. *If $V \in \mathcal{P}$, then $\mathcal{L}(V)$ is an acyclic complex, i.e., $H^*(\mathcal{L}(V)) = 0$.*

Proof. Let $\xi \in \Theta$ be nonzero. The geometric fiber of $H^j(\mathcal{L}(V))$ at the corresponding point $\bar{\xi}$ is equal to the j-th cohomology group of the complex of vector bundles

$$L_{\bar{\xi}}(V) = \{\cdots \xrightarrow{\xi} V_{j-1} \xrightarrow{\xi} V_j \xrightarrow{\xi} V_{j+1} \xrightarrow{\xi} \dots\}. \tag{28}$$

Since V is a free module, the complex $L_{\bar{\xi}}(V)$ is exact, hence, the fiber of $H^j(\mathcal{L}(V))$ vanishes at all points $\bar{\xi} \in \mathbb{P}$. Hence, $H^j(\mathcal{L}(V)) = 0$. □

4.5. Let $L_{\mathcal{D}}(V)$ be the image of the complex $\mathcal{L}(V)$ in $\mathcal{D}^b(\mathsf{Sh})$. The definition of $\mathcal{M}^0(\Lambda)$ (see 1.12) and Lemma 3.5 imply that the map $V \mapsto L_{\mathcal{D}}(V)$ extends to the functor $\Phi_{\mathcal{D}} : \mathcal{M}^b(\Lambda) \longrightarrow \mathcal{D}^b(\mathsf{Sh})$.

Theorem. *The functor $\Phi_{\mathcal{D}}$ defines an equivalence of categories.*

4.6. The scheme of the proof. Proof consists of several practically independent steps, each step proving an equivalence of some categories. Let us introduce all necessary categories.

a) $\mathcal{D}^b(\mathsf{Sh})$, the derived category of the category of coherent sheaves on $\mathbb{P}$.

b) $\mathcal{M}^b(\Lambda)$, the category of finite dimensional graded Λ-modules and $\mathcal{D}^b(\Lambda)$ the corresponding derived category.

c) Let $X = \Theta^*$ and $S = S^\bullet(X)$. Denote by $\mathcal{M}^b(S)$ the category of finitely generated graded S-modules and by $\mathcal{D}^b(S)$ the corresponding derived category.

d) Let $\mathcal{I}$ be the full subcategory of $\mathcal{D}^b(\Lambda)$ consisting of complexes isomorphic (in $\mathcal{D}^b(\Lambda)$) to complexes

$$\dots \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \dots \tag{29}$$

where $F_i \in \mathcal{P}$, i.e., F_i is a free Λ-module with finite number of generators (for all i).

e) Let $\mathcal{F}$ be the full subcategory of $\mathcal{D}^b(\mathsf{Sh})$ consisting of complexes isomorphic (in $\mathcal{D}^b(S)$) to complexes

$$\dots \longrightarrow M_{-1} \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \dots \tag{30}$$

where all the M_i are finite dimensional graded S-modules.

4.7. The scheme of the proof of Theorem 3.5: prove the following four statements:

a) The triangulated categories $\mathcal{D}^b(\Lambda)$ and $\mathcal{D}^b(S)$ are equivalent.

b) $\mathcal{I}$ and $\mathcal{F}$ are thick subcategories of $\mathcal{D}^b(\Lambda)$ and $\mathcal{D}^b(S)$, respectively (see Appendix, §5) and the corresponding quotient categories $\mathcal{D}^b(\Lambda)/\mathcal{I}$ and $\mathcal{D}^b(S)/\mathcal{F}$ are equivalent (this equivalence being the factorization of the equivalence of the previous heading).

c) The categories $\mathcal{D}^b(\Lambda)/\mathcal{I}$ and $\mathcal{M}^0(\Lambda)$ are equivalent.

d) The categories $\mathcal{D}^b(S)/\mathcal{F}$ and $\mathcal{D}^b(\mathsf{Sh})$ are equivalent.

Among these four statements a) and b) are more difficult (Theorem 5.7 and Theorem 6.2, respectively). Statement b) is easily derived from a) (Corollary 5.15). Statement d) is an easy corollary of Serre's theorem describing coherent sheaves on $\mathbb{P}$: we will prove it in the following subsection.

4.8. Proposition. (i) *$\mathcal{F}$ is a thick subcategory of the triangulated category $\mathcal{D}^b(S)$, so we may consider the triangulated quotient category $\mathcal{D}^b(S)/\mathcal{F}$.*

(ii) *$\mathcal{D}^b(S)/\mathcal{F} \simeq \mathcal{D}^b(\mathsf{Sh})$ as triangulated categories.*

Proof. Let $\mathcal{S}$ be the full subcategory of $\mathcal{M}^b(S)$ consisting of finite-dimensional S- modules and $\mathcal{M}^b(S)/\mathcal{S}$ the quotient category in the sense of Serre (see [11]). Then $\mathcal{M}^b(S)/\mathcal{S}$ is abelian. By definition the objects of $\mathcal{F}$ are isomorphic (in $\mathcal{D}^b(S)$) to complexes of modules from $\mathcal{S}$. This immediately implies that $\mathcal{F}$ is a thick subcategory in $\mathcal{D}^b(S)$ and the quotient category $\mathcal{D}^b(S)/\mathcal{F}$ is equivalent to the bounded derived category $\mathcal{D}^b(\mathcal{M}^b(S)/\mathcal{S})$. Further, due to Serre's theorem (see [S] or [Ha2], Ex. 11.5.9]) the category Sh of coherent sheaves on $\mathbb{P}$ is equivalent to $\mathcal{M}^b(S)/\mathcal{S}$. Therefore

$$\mathcal{D}^b(S)/\mathcal{F} \simeq \mathcal{D}^b(\mathcal{M}^b(S)/\mathcal{S}) \simeq \mathcal{D}^b(\mathsf{Sh}), \tag{31}$$

and we are done. □

5. Addenda to Main Theorem

5.1. In this section we will give several corollaries and addenda to Theorem 3.5. First of all, let us derive from Theorem 3.5 the main theorem (2.6).

Let us assign to each sheaf $\mathcal{E}$ on $\mathbb{P}$ the complex $c(\mathcal{E}) \in \mathcal{C}^b(\mathsf{Sh})$ whose zeroth term is equal to $\mathcal{E}$, the other terms being zero. Let $I(\mathcal{E}) \in \mathcal{D}^b(\mathsf{Sh})$ be the element of the derived category corresponding to this complex. Then, as is known (Proposition A.8), the functor I identifies the category Sh with the full subcategory of $\mathcal{D}^b(\mathsf{Sh})$ consisting of objects Y such that $H^i(Y) = 0$ for $i \neq 0$. Hence, by Theorem 3.5 the category Sh is equivalent to the subcategory of $\mathcal{M}^0(\Lambda)$ consisting of modules V such that $H^i(\mathcal{L}(V)) = 0$ for $i \neq 0$. The following simple lemma (we skip its proof) implies Theorem 2.6.

Lemma. *Let $V \in \mathcal{M}^b(\Lambda)$ be a module such that $H^i(\mathcal{L}(V)) = 0$ for $i \neq 0$ and let $H^0(\mathcal{L}(V))$ be a locally free sheaf (i.e., bundle). Then V is a nice module i.e., $H^i(L_{\bar{\xi}}(V)) = 0$ for any $i \neq 0$ and any nonzero $\xi \in \Theta$.*

5.2. The equivalence of categories $\Phi_{\mathcal{D}} : \mathcal{M}^0(\Lambda) \longrightarrow \mathcal{D}^b(\mathsf{Sh})$ defines on $\mathcal{M}^0(\Lambda)$ the structure of a triangulated category. This structure is described as follows.

a) The translation functor T. Let $V \in \mathcal{M}^b(\Lambda)$. Let $V' = V \otimes \Lambda$ considered with the natural embedding $V \longrightarrow V'$ given by $v \mapsto v \otimes 1$ and set $T(V) = V'/V$. Clearly, T defines a functor in $\mathcal{M}^0(\Lambda)$. It is easy see that if

$$0 \longrightarrow V \longrightarrow F \longrightarrow \tilde{V} \longrightarrow 0 \tag{32}$$

is an exact sequence in $\mathcal{M}^b(\Lambda)$ and $F \in \mathcal{P}$, then in $\mathcal{M}^0(\Lambda)$ there is a canonical isomorphism $\tilde{V} \simeq T(V)$.

b) Fixed triangles. Let $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ be an exact sequence in $\mathcal{M}^b(\Lambda)$, then morphisms $V' \longrightarrow V \longrightarrow V''$ (more precisely, their images in $\mathcal{M}^0(\Lambda)$) enter a fixed triangle in $\mathcal{M}^0(\Lambda)$, so all the pairs of morphisms entering the fixed triangles (up to isomorphism) are obtained.

The proof of these statements may be done simultaneously with the proof of Theorem 3.5. It is very good exercise to give a straightforward verification of all the axioms of triangulated category.

Note, that to a fixed triangle $V' \longrightarrow V \longrightarrow V''$ in $\mathcal{M}^0(\Lambda)$ consisting of nice modules, there corresponds the exact sequence

$$0 \longrightarrow \Phi(V') \longrightarrow \Phi(V) \longrightarrow \Phi(V'') \longrightarrow 0 \tag{33}$$

of bundles on $\mathbb{P}$.

5.3. Let V and W be two nice Λ-modules, $\Phi(V)$ and $\Phi(W)$ the corresponding locally free sheaves. Let us calculate the groups $\mathrm{Ext}^i(\Phi(V), \Phi(W))$, where Ext is taken in the category Sh of coherent sheaves on $\mathbb{P}$, in terms of modules V, W.

Proposition. (i) *For $i \geq 1$, we have*

$$\mathrm{Ext}^i(\Phi(V), \Phi(W)) = \mathrm{Ext}^i_{\mathcal{M}^b(\Lambda)}(V, W). \tag{34}$$

(ii) $\mathrm{Hom}(\Phi(V), \Phi(W)) = \mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(1, (V^* \otimes W)_{min})$, *where $(V^* \otimes W)_{min}$ is the unique reduced Λ-module $\mathcal{P}$-equivalent to $V^* \otimes W$.*

Proof. (i) Since V and W are nice, in the category $\mathcal{D}^b(\mathsf{Sh})$, there are isomorphisms

$$\{\dots \longrightarrow 0 \longrightarrow \Phi(V) \longrightarrow 0 \longrightarrow \dots\} \simeq \Phi_{\mathcal{D}}(V) \tag{35}$$

Therefore (see [Ha1], ch.1, sec. 6)

$$\operatorname{Ext}^i(\Phi(V), \Phi(W)) = \operatorname{Hom}_{D^b(\mathsf{Sh})}(\Phi_D(V), T^i\Phi_D(W)) \tag{36}$$

(here T is the translation functor in $D^b(\mathsf{Sh})$. By Theorem 3.5 we have

$$\operatorname{Hom}_{D^b(\mathsf{Sh})}(\Phi_D(V), T^i\Phi_D(W)) = \operatorname{Hom}_{M^0(\Lambda)}(V, T^iW). \tag{37}$$

Thus, to prove proposition 4.3 it suffices to prove the following lemma.

5.4. Lemma. *Let V and W be two Λ-modules. There is a natural isomorphism*

$$\operatorname{Hom}_{M^0(\Lambda)}(V, T^iW) \cong \operatorname{Ext}^i_{M^b(\Lambda)}(V, W) \tag{38}$$

for any $i \geq 1$.

Proof of Lemma. Let

$$0 \longrightarrow W \longrightarrow F_0 \xrightarrow{\alpha_1} F_1 \xrightarrow{\alpha_2} \dots \tag{39}$$

be an injective resolution of W. On the one hand, the definition of the translation functor T in $\mathcal{M}^0(\Lambda)$ (see 4.2) easily implies that $T^iW = \operatorname{Im}\alpha_i = \operatorname{Ker}\alpha_{i+1}$ for $i \geq 1$.

On the other hand, consider the induced sequence of Hom's in $\mathcal{M}^b(\Lambda)$

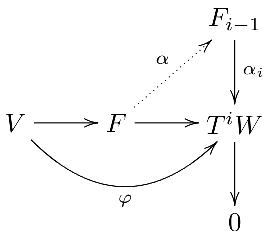
$$\dots \longrightarrow \operatorname{Hom}(V, F_{i-1}) \xrightarrow{\alpha_i^*} \operatorname{Hom}(V, F_i) \xrightarrow{\alpha_{i+1}^*} \dots \tag{40}$$

By definition of Ext we have

$$\operatorname{Ext}^i_{M^b(\Lambda)}(V, W) = \operatorname{Ker}\alpha_{i+1}^* / \operatorname{Im}\alpha_i^*. \tag{41}$$

Clearly, $\operatorname{Ker}\alpha_{i+1}^* = \operatorname{Hom}_{M^b(\Lambda)}(V, T^iW)$. Therefore, to prove Lemma, it suffices to verify that $\operatorname{Im}\alpha_i^* = \{\varphi \in \operatorname{Hom}_{M^b(\Lambda)}(V, T^iW)\}$ factors through a free module first, any morphism $\varphi \in \operatorname{Im}\alpha_i^*$ factors through a free module F_{i-1}. Further, suppose $\varphi\colon V \longrightarrow T^iW$ factors through a free module F. Consider the diagram

(42)



Since the sequence (4.1) is exact, the column of this diagram is exact and the projectivity of F implies the morphism $d : F \longrightarrow F_{i-1}$ making the diagram commutative. Hence, $\varphi \in \operatorname{Im}\alpha_i^*$. Lemma is proved. □

□

Let 1 be the trivial one-dimensional Λ-module.

Corollary. *For any nice Λ-module V, we have*

a) $H^0(\mathbb{P}, \Phi(V)) = \operatorname{Hom}_{M^b(\Lambda)}(1, V_{min})$, where V_min is the unique reduced Λ-module $\mathbb{P}$-equivalent to V.

b) $H^i(\mathbb{P}, \Phi(V)) = \operatorname{Hom}_{M^0(\Lambda)}(1, T^iW) = \operatorname{Ext}_{M^b(\Lambda)}(1, V) = H^i(\Lambda, V)$ for $i \geq 1$.

Proof. a) follows from the easily verified equality

$$\operatorname{Hom}_{M^0(\Lambda)}(1, V) = \operatorname{Hom}_{M^b(\Lambda)}(1, V_{min}) \tag{43}$$

b) follows from Proposition. □

5.5. Let $A^i(P)$ be the group of classes of cycles on $\mathbb{P}$ modulo rational equivalence, $A(P) = \oplus A^i(\mathbb{P})$ the Chow ring of $\mathbb{P}$. For any bundle $\mathcal{E}$ on $\mathbb{P}$, denote by $C_i(\mathcal{E}) \in A^i(\mathbb{P})$ its i-th Chern class and by $c_t(\mathcal{E}) = 1 + \sum c_i\mathcal{E}t^i$ the corresponding Chern polynomial (t is a formal variable), see, e.g., [Ha2], App. A.3. Let $h \in A^1(p)$ be the class of a hypersurface in $\mathbb{P}$. Recall that

$$A(p) = \mathbb{Z}[h]/(h^{n+1}) \tag{44}$$

and

$$c_t(\mathcal{O}(1)) = 1 + ht. \tag{45}$$

Proposition. *Let $V = \oplus V_j$ be a nice module. Then*

$$c_t(\Phi(V)) = \prod_i (1 + jht)^{(-1)^j \dim V_j}. \tag{46}$$

The proof of this proposition follows immediately from (4.2) and multiplicativity of $C_t(\mathcal{E})$ in exact sequences (see [Ha2], App, A.3).

5.6. Suppose $\mathcal{E}$ is a bundle on $\mathbb{P}$. Due to the main theorem $\mathcal{E} = \Phi(V)$ for a nice reduced module V, and V is recovered from $\mathcal{E}$ uniquely up to isomorphism.

Let us try to recover V from $\mathcal{E}$.

First consider the case when $\mathcal{E}$ is ample, i.e., the fibers of $\mathcal{E}$ are generated at each point by global sections of $\mathcal{E}$ over $\mathbb{P}$. Then the same is true for $\mathcal{E}(1)$. In that case the corresponding Λ-module V may be constructed explicitly. Consider linear spaces $L_0 = H^0(\mathbb{P}, \mathcal{E})$ and $L_1 = H^0(\mathbb{P}, \mathcal{E}(1))$. Let

$$V^{(0)} = \mathrm{Hom}(\Lambda, L_0), \quad V^{(1)} = \mathrm{Hom}(\Lambda, L_1). \tag{47}$$

The spaces $V^(0)$ and $V^(1)$ are naturally endowed with the Λ -module structure by the formula $\xi\varphi(\lambda) = \varphi(\xi\lambda)$, where $\xi \in \Theta$, $\lambda \in \Lambda$ and $\varphi \in V^{(0)}$ or $\varphi \in V^{(1)}$. In $V^(i)$, where $i = 0, 1$, let us introduce the grading setting

$$V_j^{(i)} = \mathrm{Hom}(\Lambda_{-i-j}, L_i). \tag{48}$$

Thus, $V^(i)$ are free graded Λ-modules, where $V^{(0)}$ is the direct sum of $\dim L_0$ modules $\Lambda[-n-1]$ and $V^{(1)}$ is the direct sum of $\dim L_1$ modules $\Lambda[-n-2]$. In Θ, choose a basis $\xi_0, \dots, \xi_n$. Let $x_0, ..., x_n$ be the dual basis of $X = \Theta^* = H^0(\mathbb{P}, \mathcal{O}(1))$. Each element x_l defines the linear mapping $x_l \colon L_0 \longrightarrow L_1$. Define the Λ-module morphism $\Delta \colon V^{(0)} \longrightarrow V^{(1)}$ by the formula

$$(\Delta\varphi)(\lambda) = \sum_{0 \le l \le n} x_i\varphi(\xi_l\lambda), \text{ for } \varphi \in V^{(0)}, \quad \lambda \in \Lambda. \tag{49}$$

Set $V = \mathrm{Ker}\,\Delta$.

Proposition. *V is a nice reduced Λ-module and $\Phi(V) \cong \mathcal{E}$. (we assume that $\mathcal{E}$ is ample).*

The proof of this Proposition is easily derived from Serre's theorem and the proof of Theorem 7 of the following section (see in particular the constructions in the proof of Lemma 5.9).

5.7. It seems that in general case it is quite difficult to construct module V such that $\Phi(V) = \mathcal{E}$. However, we may find dimensions $\dim V_j$ of homogeneous components of V. In the following subsections we will consider a more general situation.

Let Z be an object of $D^b(\mathsf{Sh})$. Due to Theorem 3.5 $Z = \Phi_D(V)$ for a unique (up to isomorphism) reduced Λ-module V. The following theorem answers how to find dimensions of homogeneous components of V.

Let Ω^k be the bundle of k -forms on $\mathbb{P}$. There is the exact sequence of bundles

$$0 \longrightarrow \Omega^k(k) \longrightarrow \Lambda^k(X) \longrightarrow \Omega^{k-1}(k) \longrightarrow 0, \tag{50}$$

where $\Lambda^k(X)$ is the constant bundle, its fiber being the k-th exterior power of X (so that its dimension is C^k_{n+1}. If we consider each term of this ex act sequence as the complex of sheaves consisting of one term, i.e., as the element of $D^b(\mathsf{Sh})$ we obtain the triangle

$$\Omega^k(k) \longrightarrow \Lambda^k(X) \longrightarrow \Omega^{k-1}(k) \longrightarrow T\Omega^k(k) \tag{51}$$

in $D^b(\mathsf{Sh})$. Since the triangle consists of locally free sheaves, it may be tensored by Z and by $\mathcal{O}(-j-k)$ to obtain

$$Z \otimes \Omega^k(-j) \longrightarrow Z \otimes \Lambda^k(X)(-j-k) \longrightarrow Z \otimes \Omega^{k-1}(-j) \longrightarrow Z \otimes T\Omega^{k-1}(-j) \tag{52}$$

Denote by $\mathbb{H}^l(\mathbb{P},\cdot)$ the hyperhomology group corresponding to elements of this triangle. Consider the connecting homomorphism

$$\delta\colon \mathbb{H}^{j+k-1}(\mathbb{P}, Z\otimes\Omega^{k-1}(-j) \longrightarrow \mathbb{H}^{j+k}(\mathbb{P}, Z\otimes\Omega^{k}(-j)). \tag{53}$$

5.8. Theorem. *Let $Z=\Phi_D(V)$ for the reduced Λ-module V. Then*

$$\dim V_j = \sum_{0\le k\le n} \dim\{\mathbb{H}^{j+k}(\mathbb{P}, Z\otimes\Omega^{k}(-j)/\delta\mathbb{H}^{j+k-1}(\mathbb{P}, Z\otimes\Omega^{k-1}(-j))\} \tag{54}$$

Proof. Let $\Lambda[-k]$ be the free module with one generator of degree $-k$ and $T_k\subset\Lambda[-k]$ the submodule generated by the elements of $\Lambda[-k]$ of positive degree and $Q_k=\Lambda[-k]/T_k$ (see 2.7). The elements of degree 1 form the Λ-submodule $R_k\subset Q_k$ of dimension C^k_{n+1}. It is clear that $Q_k/R_k\cong Q_{k-i}[-1]$ so that there is the exact sequence of Λ-modules

$$0\longrightarrow R_k\longrightarrow \mathrm{Q}_k \xrightarrow{\alpha} Q_{k-1}[-1]\longrightarrow 0. \tag{55}$$

Set $V_{j,k}=\{v\in V_j, \lambda v=0 \text{ for any } \lambda \text{ such that } \deg\lambda>k\}$

The subspaces $V_{j,k}$ define the decreasing filtration of V_j so that $V_{j,n}=V_j$ since V is a reduced Λ-module (it is the single place where we use that V is reduced). Thus,

$$V_j=V_{j,n}\supset V_{j,n-1}\supset\cdots\supset V_{j,0}\supset V_{j,-1}=\{0\}. \tag{56}$$

and

$$\dim V_j=\sum_{0\le k\le n}\dim(V_{j,k}/\dim V_{j,k-1}). \tag{57}$$

Further, there is the natural isomorphism

$$V_{j,k}\cong \mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(Q_k[j+k],V) \tag{58}$$

(an element $v\in V_{j,k}$ corresponds to the morphism φ that transforms the generator of $Q_k[j+k]$ into v). It is easy to verify that the subspace $V_{j,k-1}\subset V_{j,k}$ corresponds to morphisms $\varphi\colon Q_k[j+k]\longrightarrow V$ of the form $\varphi=\varphi\_1circ\alpha[j+k]$ for some

$$\varphi_1\colon Q_{k-1}[j+k-1]\longrightarrow V. \tag{59}$$

Therefore there is the isomorphism

$$V_{j,k}/V_{j,k-1}=\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(Q_k[j+k],V)/\alpha^*\,\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(Q_{k-1}[j+k-1],V). \tag{60}$$

Now let us pass in the right-hand side of this equality to morphisms in the category $\mathcal{M}^0(\Lambda)=\mathcal{M}^b(\Lambda)/\mathcal{P}$. If, to each morphism of $\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(Q_k[j+k],V)$, we assign its class in $\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_k[j+k],V)$, we obtain, for any k such that $0\le k\le n$, the homomorphism

$$\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(Q_k[j+k],V)/\alpha^*\,\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(Q_{k-1}[j+k-1],V)\xrightarrow{r}$$
$$\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_k[j+k],V)/\alpha^*\,\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_{k-1}[j+k-1],V).$$

□

5.9. Lemma. *The homomorphism r is isomorphism.*

Proof. Clearly, r is epimorphism. To prove its injectivity it suffices to show that if W is an arbitrary reduced Λ-module and the morphism $\varphi\colon Q_k\longrightarrow W$ factorizes through a free Λ-module F, i.e.,

$$\varphi\colon Q_k\xrightarrow{\psi} F\xrightarrow{\theta} W \tag{61}$$

then φ also factorizes through $Q_{k-1}[-1]$, i.e.,

$$\varphi\colon Q_k\xrightarrow{\alpha} Q_{k-1}[-1]\longrightarrow W \tag{62}$$

To prove this, note that φ presents in the form (4.8) if and only if $\varphi(q)=0$ for any $q\in Q_k$ such that $\deg q=0$. Further, F in (4.7) may be assumed free with one generator of degree l, so that $F=\Lambda[l]$ The non-trivial morphism $\varphi\colon Q_k\longrightarrow\Lambda[l]$ exists only for $-n-k-1\le l\le -n-1$ and by the above ψ factorizes through $Q^{[}_{k-1}-1]$ for $l<-n-1$. Therefore it suffices to consider the case $F=\Lambda[-n-1]$. Let e be generator of $\Lambda[-n-1]$ and $w=\theta(e)\in W$ (in 4.7). Since $\varphi\neq 0$, we have $W-n-1\neq 0$. Since W is reduced, $W_0=\{0\}$, and hence $\varphi(q)=0$ for any $q\in Q_k$ such that $\deg q=0$. Therefore φ factorizes through $Q^{[-1]}_{k-1}$ and Lemma is proved. □

5.10. Thus, eqs. (4.5) and (4.6) imply

(63) $$\dim V_j = \sum_{0\le k\le n} \dim\{\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_k[j+k], V)/\alpha^* \,\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_{k-1}[j+k-1], V)\}.$$

Now, using main Theorem 3.5 we may express the dimensions of the right-hand side of this equality in terms of the object Z. For the sake of further references, let us formulate the property (vii) of subsect. 2.7.

5.11. Lemma. *Let V be a nice module. Then so is $\tilde{V} = TV[-1]$, and we have $\Phi(V) = \Phi(\tilde{V}) \otimes \mathcal{O}(1)$.*

5.12. The category $\mathcal{M}^0(\Lambda)$ is triangulated. To the exact sequence (4.4) the triangle in $\mathcal{M}^0(\Lambda)$

(64) $$R_k \longrightarrow Q_k \longrightarrow Q_{k-1}[-1]$$

corresponds (see remark in 4.2). Therefore, in $\mathcal{M}^0(\Lambda)$, there exists a triangle (axiom TR2, see Appendix, Section 1)

(65) $$T^{-1}Q_{k-1}[-1] \longrightarrow R_k \longrightarrow Q_k$$

All modules in this triangle are nice (Lemma 4.11). Due to Remark in 4.2 to these modules there corresponds the exact sequence of bundles on $\mathbb{P}$:

(66) $$0 \longrightarrow \Phi(T^{-1}Q_{k-1}[-1]) \longrightarrow \Phi(R_k) \longrightarrow \Phi(Q_k) \longrightarrow 0$$

Clearly, $\Phi(R_k)$ is a trivial bundle with the fiber $\Lambda^k(\Theta)$. Further, Q_k is a nice module and $Phi(Q_k) = \Omega^{n-k}(n-k+1)$, (see 2.7 (vi)). Due to Lemma 4.11

(67) $$\Phi(T^{-1}Q_{k-1}[-1]) = \Phi(Q_{k-1}[-1]) \otimes \mathcal{O}(-1) = \Omega^{n-k+1}(n-k+1).$$

Hence, the exact sequence (4.9) is of the form

(68) $$0 \longrightarrow \Omega^{n-k+1}(n-k+1) \longrightarrow \Lambda^k(\Theta) \longrightarrow \Omega^{n-k}(n-k+1) \longrightarrow 0.$$

It is easy to see that the sequence (4.10) is dual to the exact sequence (4.3)

(69) $$O \longrightarrow \Omega^k(k) \longrightarrow \Lambda^k(X) \longrightarrow \Omega^{k-1}(k) \longrightarrow 0$$

considered at the beginning of this subsection.

5.13. Lemma. *Let $1[s]$, where $s > 0$, be the one-dimensional Λ-module with one generator of degree s and V an arbitrary Λ-module. Then*

(70) $$\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(1[s], V) = \mathbb{H}^s(\mathbb{P}, \Phi_D(V) \otimes \mathcal{O}(-s))$$

where $\mathbb{H}^s$ is the s-th hyperhomology group of $\Phi_D(V)$.

The proof follows immediately from the definition of

(71) $$\mathbb{H}^s(\mathbb{P}, Z) = \mathrm{Hom}_{D^b(\mathsf{Sh})}(1, T^s Z), \quad \text{where 1 is the trivial bundle,}$$

Theorem 3.5 and Lemma 5.11.

Note that statements of corollary in 4.4 are special cases of this lemma (for nice Λ-modules).

5.14. Due to Lemma 4.13 we have

$$\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_k[k+j], V) = \mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(\mathbb{P}, Q_k^*[k+j] \otimes V) =$$
$$\mathbb{H}^{k+j}(\mathbb{P}, \Phi_D(Q_k^* \otimes V)(-j-k)) = \mathbb{H}^{k+j}(\mathbb{P}, Z \otimes \Omega^k(-j));$$

similarly,

(72) $$\mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_{k-1}[j+k-1] \otimes V) = \mathbb{H}^{k+j-1}(\mathbb{P}, Z \otimes \Omega^{k-1}(-j)).$$

The homomorphism

(73) $$\varphi\colon\ \mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_{k-1}[j+k-1], V) \longrightarrow \mathrm{Hom}_{\mathcal{M}^0(\Lambda)}(Q_k[k+j], V)$$

corresponds to the connecting homomorphism

(74) $$\delta^*\colon \mathbb{H}^{k+j-1}(\mathbb{P}, Z \otimes \Omega^{k-1}(-j)) \longrightarrow \mathbb{H}^{k+j}(\mathbb{P}, Z \otimes \Omega^k(-j)).$$

for the exact sequence obtained from (4.3) after tensoring by Z and by $\mathcal{O}(-j-k)$. Theorem 4.8 is proved.

5.15. Corollary. *Suppose $\mathcal{E}$ is a bundle on $\mathbb{P}$ such that $\mathcal{E} = \Phi(V)$ for a nice reduced Λ-module $V = \oplus V_j$. Then*

(75) $$\dim V_j = \sum_{0\le k\le n} \dim\{\mathbb{H}^{k+j}(\mathbb{P}, \mathcal{E} \otimes \Omega^k(-j))/\delta^*\mathbb{H}^{k+j-1}(\mathbb{P}, \mathcal{E} \otimes \Omega^{k-1}(-j))\}$$

5.16. Corollary. *Let $V = \oplus V_j$ be a nice reduced Λ-module. Then V_j may be non-zero only for $-n \le j \le n$.*

The proof follows immediately from Corollary 5.15.

6. Equivalence of the derived categories

6.1. Suppose $X = \Theta^*$ is the space dual to Θ. Denote by $S = S(X)$ the symmetric algebra of the space X. On $S(X)$ define the usual grading setting $\deg x = 1$ for $x \in X$, We will often identify $S(X)$ with the algebra of polynomial functions on Θ.

The method initiated by Serre consists in the reduction of the study of the coherent sheaves of modules on $P = P(\Theta)$ to the study of graded modules over $S(X)$.

The objective of this section is to prove the algebraic theorem which links the category of graded Λ-modules and the category of graded S-modules (Theorem 5.7). It is the algebraic base of the geometric theorem 2.6.

To formulate and prove Theorem 5.7 we need some notions and results of derived categories. The main facts on derived categories are contained in [Ha1] and [V]; for convel1ience we put formulations of several results in Appendix.

6.2. Thus, let $\Lambda = \Lambda(\Theta)$ be the exterior algebra of Θ and $\Lambda_j \subset \Lambda$ the space of homogeneous elements of degree j where $0 \le j \le n+1$. Recall, that $M(\Lambda)$ is the category of left Graded Λ-modules, where morphisms are the morphisms of degree 0 of graded Λ-modules, and $M^b(\Lambda)$ is the full subcategory of $M(\Lambda)$ consisting of finitely generated (= finite dimensional over Λ) modules.

Further, let $S = S(X)$ be the symmetric algebra of $X = \Theta^*$ where $S_j \subset S$ is the space of homogeneous elements of degree j, where $j \ge 0$. Denote by $\mathcal{M}(S)$ the category of graded S- modules, where morphisms are the morphisms of degree 0 of graded S-modules. Let $\mathcal{M}^b(S)$ be the full subcategory of $\mathcal{M}(S)$ consisting of finitely generated S -modules.

6.3. Let us introduce the categories of complexes of Λ and S-modules needed in what follows.

The complex of Λ-modules is the sequence

$$\ldots \longrightarrow V^{(i-1)} \xrightarrow{\partial_{i-1}} V^{(i)} \xrightarrow{\partial_i} V^{(i+1)} \longrightarrow \ldots \tag{76}$$

of modules of $\mathcal{M}(\Lambda)$ where $\partial_i \colon V^{(i)} \longrightarrow V^{(i+1)}$ are linear mappings of degree 0 of graded spaces *anticommuting* with the action of any $\xi \in \Theta$ (i.e., $\partial_i\xi = -\xi\partial_i$) satisfying $\partial_{i+1}\partial_i = 0$. (Thus, $\partial_i = 0$ is not a morphism in $\mathcal{M}(\Lambda)$. A set of morphisms $f_i \colon V^{(i)} \longrightarrow \widetilde{V^{(i)}}$ in $\mathcal{M}(\Lambda)$, such that $f_{i+1}\partial_i = \tilde{\partial}_i f_i$ for any i, is called *a morphism of the complex* $\{V^{(i)}, \partial_i\}$ *into the complex* $\{\widetilde{V^{(i)}}, \tilde{\partial}_i\}$. Let $C(\Lambda)$ be the category of all complexes of Λ-modules with the described morphisms.

Let $C^b(\Lambda)$ be the complete subcategory of $C(\Lambda)$ consisting of bounded complexes of finite dimensional modules. In other words, $\{\widetilde{V^{(i)}}, \tilde{\partial}_i\} \in C^b(\Lambda)$ if $V^{(i)} \in \mathcal{M}^b(\Lambda)$ for any i and $V^{(i)} = \{0\}$ for all i but a finite number.

Let us introduce now categories of complexes of S- modules. A complex of S- modules is a sequence

$$\ldots \longrightarrow M^{(i-1)} \xrightarrow{d_{i-1}} M^{(i)} \xrightarrow{d_i} M^{(i+1)} \longrightarrow \ldots \tag{77}$$

of modules of $\mathcal{M}(S)$, where $d_i \colon W^{(i)} \longrightarrow W^{(i+1)}$ are morphisms of $\mathcal{M}(S)$ such that $d_{i+1}d_i = 0$ (thus, d_i *commute* with the S-action.

A set of morphisms $f_i \colon W^{(i)} \longrightarrow W^{(i+1)}$ in $\mathcal{M}(S)$ such that $f_{i+1}d_i = \tilde{d}_i f_i$ for any i is called a *morphism of the complex* $\{W^{(i)}, d_i\}$ *into the complex* $\{\widetilde{W}^{(i)}, \tilde{d}_i\}$. Let $C(S)$ be the category of all complexes of S-modules with the just described morphisms.

Denote by $C^b(S)$ the subcategory of $C(S)$ consisting of bounded complexes of finitely generated modules. In other words, $\{W^{(i)}, d_i\} \in C^b(S)$ if all $H_i W^{(i)}$ are finitely generated S-modules and $W^{(i)} = \{0\}$ for any i but a finite number.

6.4. It will be often convenient to describe complexes $\{V^{(i)}, \partial_i\} \in C(\Lambda)$ as follows. Let $V_j^{(i)}$ be the space of homogeneous elements of degree j in $V^{(i)}$. Consider the bigraded space $V = \underset{i,j}{\oplus} V_j^{(i)}$. On this space the algebra Λ and the linear operator $\partial = \oplus\, \partial_j$ act so that the following conditions hold:

(i) The operator of multiplication by $\xi \in \Theta$ is of bidegree (0, 1), i.e., $\xi(V_j^{(i)}) \subset V_{j+1}^{(i)}$.

(ii) The operator ∂ has the bidegree $(1, 0)$, i.e., $\partial(V_j^{(i)}) \subset V_j^{(i+1)}$;

(iii) $\xi\partial = -\partial\xi$ for $\xi \in \Theta$;

(iv) $\partial^2 = 0$.

The morphism $\{f_i\}\colon \{V^{(i)}, \partial_i\} \longrightarrow \{\widetilde{V^{(i)}}, \tilde{\partial}_i\}$ of complexes of Λ-modules defines a bigraded Λ-module morphism $V \longrightarrow \tilde{V}$ of bidegree $(0,0)$, i.e., $f(V_j^{(i)}?V_j^{(i)}$, so that $f\partial = \tilde{\partial} f$.

It is easy to see that the category $C(\Lambda)$ is equivalent to the category of bigraded Λ-modules V endowed with the operator $\partial\colon V \longrightarrow V$ satisfying (i)–(iv). A module V corresponds to a complex of $C^b(\Lambda)$ if and only if $\dim V < \infty$.

Similarly, the category $C(S)$ is equivalent to the category of bigraded S-modules $W = \bigoplus_{i,j} W^{i,j}$ endowed with the linear operator $d\colon W \longrightarrow W^{i,j}$ such that

(i)′ the $x \in X \subset S$ have the bidegree $(0,1)$;

(ii)′ d has the bidegree $(1,0)$;

(iii)′ $sd = ds$, where $s \in S$;

(iv)′ $d^2 = 0$.

The morphism $\{f_i\}\colon \{W^{(i)}, d_i\} \longrightarrow \{\widetilde{W^{(i)}}, \tilde{d}_i\}$ of the complex of S-modules defines the morphism $f\colon W \longrightarrow \widetilde{W}$ of the bigraded S-modules of the bidegree (0, 0) such that $fd = \tilde{d}f$.

Under the above identification $\{W^(i), d_i\} \longleftrightarrow W$, the complexes from $C^b(S)$ correspond to finitely generated S-modules.

6.5. In this subsection we will define the functor $F\colon C(\Lambda) \longrightarrow C(S)$. Let $V = \bigoplus_{i,j} V_j^{(i)} \in C(\Lambda)$. Consider the space $W = S \otimes_{\mathbb{K}} V$. In W, define the S-action setting $s(s_1 \otimes v) = ss_1 \otimes v$. Define the differential $d\colon W \longrightarrow W$ by the formula

$$d(s \otimes v) = \sum x_l s \otimes \xi_l v + s \otimes \partial v, \tag{78}$$

where $\{\xi_l\}$ and $\{x_l\}$ are dual bases in Θ and X. In W define the bigrading as follows: if $j \in S_k$, and $v \in V_j^{(i)}$ then set $s \otimes v \in W_{k-j}^{(i+j)}$. It is easy to verify that the S-action in W and the operator d satisfy (i)'-(iv)' of 5.4. Therefore W defines the complex of $C(S)$. The mapping $V \mapsto W$ defines the functor $F\colon C(\Lambda) \longrightarrow C(S)$. It is clear that F commutes with translations T in categories of complexes $C(\Lambda)$ and $C(S)$. It is easy also to verify that if $V \in C^b(\Lambda)$ then $F(V) \in C^b(S)$.

6.6. A morphism $f\colon V \longrightarrow \tilde{V}$ in $C(\Lambda)$ is a quasiisomorphism (see Appendix, subsection 5) if the corresponding mapping of cohomology $f^*\colon H^*(V) \longrightarrow H^*(\tilde{V})$ is isomorphism. Denote by $D(\Lambda)$ the quotient category of $C(\Lambda)$ modulo the family of quasiisomorphisms (see Appendix, subsection 6). The category $D(S)$ is similarly defined.

Let $D^b(\Lambda)$ and $D^b(S)$ be the quotient categories of $C^b(\Lambda)$ and $C^b(S)$ modulo quasiisomorphisms. The Categories $D(\Lambda)$ and $D(S)$ are triangulated, while $D^b(\Lambda)$ and $D^b(S)$ are their full triangulated subcategories (see [V]). The main theorem of this section runs as follows.

6.7. Theorem. (i) *The functor $F\colon C^b(\Lambda) \longrightarrow C^b(S)$ extends to the functor*

$$F_D\colon D^b(\Lambda) \longrightarrow D^b(S). \tag{79}$$

(ii) *F_D defines an equivalence of triangulated categories $D^b(\Lambda)$ and $D^b(S)$.*

6.8. Lemma. (i) *The functor F is exact, commutes with T and transforms the cone of the morphism $f\colon V \longrightarrow \tilde{V}$ in the cone of the morphism $F(f)\colon F(V) \longrightarrow F(\tilde{V})$.*

(ii) *The restriction of F onto $C^b(\Lambda)$ transforms quasiisomorphisms into quasiisomorphisms.*

Proof. All the statements of the heading (i) are quite straightforward. Let us prove (ii).

Let $f\colon V \longrightarrow \tilde{V}$ be a morphism in $C^b(\Lambda)$. Denote by Z the cone of f. Clearly, $Z \in C^l(\Lambda)$. Then f is a quasiisomorphism if and only if Z is acyclic. Since $F(Z)$ is the cone of $F(f)\colon F(V) \longrightarrow F(\tilde{V})$, it suffices to prove that F transforms acyclic objects into acyclic ones.

Consider $S \otimes Z$ as a bicomplex of S-modules, setting $(S \otimes Z)^{p,q} = S \otimes Z_q^{(p)}$ with two differentials $d' = \sum x_l \otimes \xi_l$ and $d'' = 1 \otimes \partial$ of bidegree $(1,0)$ and $(0,1)$ respectively. Then the differential d in $F(Z)$ is $d = d' + d''$. Since Z is acyclic, $H(S \otimes Z) = 0$. Further, since $Z \in C^l(\Lambda)$, we have $(S \otimes Z) = 0$ for any pairs (p,q), but a finite number. Hence, it is easy to verify that $F(Z)$ is acyclic (either straightforward or using the spectral sequence for the bicomplex [4] see Theorem XI.6.1). Lemma is proved. □

Lemma implies that the functor $F\colon C^b(\Lambda) \longrightarrow C^b(S)$ extends to the functor $F_D\colon \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{D}^b(S)$, i.e., the first part of Theorem 5.7 is proved.

6.9. In the remainder of this section we prove the second part of Theorem 5.7.

Lemma. *(i) There is the functor* $G\colon C(S) \longrightarrow C(\Lambda)$ *right conjugate to* F *i.e., such that there is the isomorphism*

$$\alpha_{V,W}\colon \operatorname{Hom}(F(V), W) \xrightarrow{\sim} \operatorname{Hom}(V, G(W)), \tag{80}$$

where $V \in C(\Lambda)$, $W \in C(S)$.

(ii) The functor G *commutes with translations.*

(iii) The isomorphism $\alpha_{V,W}$ *transforms homotopic mappings into homotopic mappings (see Appendix, subsection 4).*

Proof. (i) Let us construct the functor G explicitly. Let $\{W^{(i)}, d_i\} \in C(S)$. As above, we identify $\{W^{(i)}, d_i\}$ with the bigraded S-module $W = \oplus W_i^{(i)}$. Set

$$G(W) = \operatorname{Hom}_k(\Lambda, W) \tag{81}$$

and define a bigrading in $V = G(W)$ as follows. If $\varphi(\Lambda_k) \subset W_{-j-k}^{(i+j+k)}$ for any k, then $\varphi \in V_j^{(i)}$. The Λ-action on V we will define by the formula $\xi\varphi(\lambda) = \varphi(\xi\lambda)$ for $\varphi \in V$, $\xi \in \Theta$, $\lambda \in \Lambda$. Define the differential $\partial\colon V \longrightarrow V$ by the formula

$$(\partial\varphi)(\lambda) = -\sum x_l \varphi(\xi_l \lambda) + d\varphi(\lambda) \text{ for } \varphi \in \Lambda. \tag{82}$$

To prove that

$$\alpha_{V,W}\colon \operatorname{Hom}(F(V), W) \xrightarrow{\sim} \operatorname{Hom}(V, G(W)) \tag{83}$$

is isomorphism in the most simple way we will verify that both sides are naturally identified with the space of $\mathbb{K}$-*linear* mappings $\psi\colon V \longrightarrow W$, satisfying

a) $\psi(V_j^{(i)} \subset W_{-j}^{(i-j)}$

b) $\psi(\partial v) + \sum x_l \psi(\xi_i v) = d\psi(v)$ for $v \in V$.

This simple check we left to the reader.

(ii) is clear from the construction.

(iii) Suppose $V \in C(\Lambda)$ and C_V is the cone over V (i.e., the cone of the identity morphism id_V). Then for any $\tilde V \in C(\Lambda)$ the image of the homomorphism $\operatorname{Hom}(C_V, \tilde V) \longrightarrow \operatorname{Hom}(V, \tilde V)$ induced by the natural embedding $V \longrightarrow C_V$ coincides with the subgroup

$$\operatorname{Ht}(V, \tilde V) = \{f \in \operatorname{Hom}(V, \tilde V) \mid f \text{ is homotopic to zero}\}. \tag{84}$$

By Lemma 5.8 (i) and the heading (i) of this Lemma we have

$$\begin{aligned} &\operatorname{Ht}(F(V), W) = \operatorname{Im}(\operatorname{Hom}(C_{F(V)}, W) \longrightarrow \operatorname{Hom}(F(V), W)) = \\ &\operatorname{Im}(\operatorname{Hom}(F(C_V), W) \longrightarrow \operatorname{Hom}(F(V), W)) = \\ &\operatorname{Im}(\operatorname{Hom}(C_V, G(W))) \longrightarrow \operatorname{Hom}(V, G(W)) = \operatorname{Ht}(V, G(W)) \end{aligned} \tag{85}$$

for any $V \in C(\Lambda)$ and $W \in C(S)$. □

6.10. Lemma. *If* $V \in C^b(\Lambda)$ *and* $W \in C^b(W)$, *then*

$$\operatorname{Hom}_{D^b(S)}(F(V), W) = \operatorname{Hom}_{D(\Lambda)}(V, G(W)). \tag{86}$$

Proof. Since $V \in C^b(\Lambda)$, the complex $F(V)$ consists of projective S-modules and is bounded. Hence,

$$\operatorname{Hom}_{D^b(S)}(F(V), W) = \operatorname{Hom}_{C(S)}(F(V), W)/Ht(F(V), W) \tag{87}$$

(see Appendix, subsection 9) Similarly, the complex $G(W)$, where $W \in C^b S)$, consists of injective Λ-modules and is bounded from the right so

$$\operatorname{Hom}_{C(S)}(F(V), W)/Ht(F(V), W) = \operatorname{Hom}_{D(\Lambda)}(V, G(V)). \tag{88}$$

Lemma follows now from Lemma 5.9 (i), (iii).

□

6.11. Let $V \in C(\Lambda)$. Set $W = F(V)$. Let $\mathrm{id}_{F(V)}$ be the identity morphism $F(V) \longrightarrow F(V)$. Define $i_V \in \mathrm{Hom}_{C(\Lambda}(V, G(F(V)))$ by the formula

$$i_V : \alpha_{V,F(V)}(\mathrm{id}_{F(V)}). \tag{89}$$

It is clear that the set of morphisms $\{i_V,$ where $V \in C(\Lambda)\}$ defines the morphism of the identity functor $C(\Lambda) \longrightarrow C(\Lambda)$ in to the functor $G \circ F : C(\Lambda) \longrightarrow C(\Lambda)$.

Lemma. *If $V \in C^b(\Lambda)$ then $i_V : V \longrightarrow G(F(V))$ is quasiisomorphism.*

Proof. a) First let V be the trivial one-dimensional complex, i.e., $V^{(0)} = 1$ and $V^{(i)} = 0$ for other i. Set $U = G(F(V)) = \oplus U_j^{(i)}$. The explicit description of F and G (see 5.6, 5.9) implies that $U_j^{(i)} = \mathrm{Hom}_k(\Lambda_{j-i}, S_i)$ and the differential $\partial : U_j^{(i)} \longrightarrow U_j^{(i+1)}$ acts by the formula $(\partial\varphi)(\lambda) = -\sum x_l \varphi(\xi_l \lambda)$, where $\lambda \in \Lambda_{j-i-1}$. The morphism $i_V : V \longrightarrow U$ transforms $e \in V_0^{(0)}$ into the identity mapping in $U_0^{(0)} = \mathrm{Hom}_{\mathbb{K}}(\Lambda_0, S_0) = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{K})$.

It is easy to see that we have obtained the standard Koszul complex for $S(X)$ corresponding to the regular sequence $(x_0, \dots, x_n)$ (see [Ha2], III, 3 or [GH] ch. 5, sec.3). Hence $i_V : V \longrightarrow U$ is quasiisomorphism for the trivial one-dimensional complex.

b) The first part of the proof immediately implies that i_V is quasiisomorphism for any one-dimensional complex (i.e., for a complex such that $V^{(i_0)} = 1[j]$ for some (i_0, j), the other $V^{(i)}$ being zero). Since each complex $V \in C^b(\Lambda)$ is finite-dimensional, to complete the proof it suffices to make the induction in $\dim V$.

Let $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ be an exact sequence of complexes of $C^b(\Lambda)$ and $\max\{\dim V', \dim V''\} < \dim V$. Since $G \circ F$ is an exact functor and i is a functor morphism, we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle i_{V'}} & & \big\downarrow{\scriptstyle i_V} & & \big\downarrow{\scriptstyle i_{V''}} & & \\ 0 & \longrightarrow & G(F(V')) & \longrightarrow & G(F(V')) & \longrightarrow & G(F(V'')) & \longrightarrow & 0 \end{array} \tag{90}$$

By the inductive hypothesis, i_V and $i_{V''}$ are quasiisomorphisms. From the exact sequence of cohomology and five-lemma we immediately derive that i_V is quasiisomorphism. □

6.12. Let us now complete the proof of Theorem 5.7.

a) $F_D : \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{D}^b(S)$ is complete functor.

Let $V, \tilde{V} \in \mathcal{D}^b(\Lambda)$. Consider the sequence of mappings

$$\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(V, \tilde{V}) \xrightarrow{F_{\mathcal{D}}} \mathrm{Hom}_{\mathcal{D}^b(S)}(F(V), F(\tilde{V})) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(V, G(F(\tilde{V}))). \tag{91}$$

The composition of mappings is clearly of the form $\varphi \mapsto i_{\tilde{V}} \circ \varphi$, where $i_{\tilde{V}} \in \mathrm{Hom}\,\mathcal{D}^b(\Lambda)(V, G(F(\tilde{V})))$ is an isomorphism by Lemma 5.11, and therefore the composition is an isomorphism. On the other hand, α is an isomorphism by Lemma 5.10. Hence,

$$F_{\mathcal{D}} : \ \mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(V, \tilde{V}) \longrightarrow \mathrm{Hom}_{\mathcal{D}^b(S)}(F(V), F(\tilde{V})) \tag{92}$$

is isomorphism.

b) By a) the subcategory $F^b_{\mathcal{D}}(\Lambda))$ is a full triangulated subcategory of $\mathcal{D}^b(S)$. Besides, $F_{\mathcal{D}}(\mathcal{D}^b(\Lambda))$ contains all objects presentable by free complexes W with one generator (they are of the form $F(V)$, where $\dim V = 1$). Such objects are generators in $\mathcal{D}^b(S)$ (the theorem on syzygies), therefore $F_{\mathcal{D}}(\mathcal{D}^b(\Lambda)) = \mathcal{D}^b(S)$ and Theorem is proved.

6.13. The functor inverse to $F_{\mathcal{D}}$ may be described explicitly.

Let $\tilde{C}(\Lambda)$ be a full subcategory of $C(\Lambda)$ consisting of bounded from the left complexes V of finite dimensional modules, such that $\dim H^*(V) < \infty$. Let $\tilde{D}(\Lambda)$ be the corresponding derived category. It is easy to verify (similarly to [14], section II.1) that the embedding $C^b(\Lambda) \longrightarrow \tilde{C}(\Lambda)$ defines the equivalence of derived categories $\mathcal{D}^b(\Lambda) \longrightarrow \tilde{\mathcal{D}}(\Lambda)$.

Proposition. *(i) $G : C(S) \longrightarrow C(\Lambda)$ is the faithful functor that transforms quasiisomorphisms of $C^b(S)$ into quasiisomorphisms.*

(ii) $G(C^b(S)) \subset \tilde{C}_b(\Lambda)$.

(iii) The functor $G_{\mathcal{D}} : \mathcal{D}^b(S) \longrightarrow \tilde{\mathcal{D}}(\Lambda) \cong \mathcal{D}^b(\Lambda)$ is the inverse of $F_{\mathcal{D}}$.

Proof. (i) is proved as Lemma 5.11.

(ii). The definition of the functor G implies that if $W \in C^b(S)$ then the complex $G(W)$ is bounded from the left and consists of finite-dimensional modules. Let us show that $\dim H^*(G(W)) < \infty$. By Theorem 5.7 there is the complex $V \in C^b(\Lambda)$ such that W is quasiisomorphic to $F(V)$. By the part of (i) $G(W)$ is quasiisomorphic to $G(F(V))$. By Lemma 3.11, $H^*G(F(V))) \simeq H^*(V)$. Hence, $\dim H^*(G(W)) = \dim H^*(V) < \infty$. The heading (iii) directly follows from Theorem 5.7. □

6.14. We have in view the further applications of Theorem 5.7. Therefore concluding this section we give one more proposition.

Let $\mathcal{F}$ be a full subcategory of $\mathcal{D}^b(S)$ consisting of objects isomorphic (in $\mathcal{D}^b(S)$) to finite dimensional complexes of S-modules. On the other hand, let $\mathcal{J}$ be a full subcategory of $\mathcal{D}^b(\Lambda)$ consisting of objects isomorphic (in $\mathcal{D}^b(\Lambda)$) to complexes of free modules.

Lemma. *The complex* $W \in \mathcal{D}^b(S)$ *belongs to* $\mathcal{F}$ *if and only if* $\dim H^*(W) < \infty$.

Proof. The necessity of the lemma is obvious. Conversely, let $\dim H^*(W) < \infty$. For any number $N \in \mathbb{Z}_+$, denote by $W(N)$ the subcomplex of W generated by homogeneous elements of degree $\geq N$. Clearly, $W/W(N)$ is a finite dimensional complex. On the other hand, since $\dim H^*(W) < \infty$ we have $H^*(W(N)) = 0$ for large N. Hence, W is quasiisomorphic to $W/W(N)$, i.e., $W \in F$. □

Proposition. *Functors* F *and* G *perform equivalence of subcategories* $\mathcal{F}$ *and* $\mathcal{J}$*.*

Proof. Since F and $\mathcal{J}$ are full subcategories, by Theorem 5.7 it suffices to verify that $F_{\mathcal{D}}(\mathcal{J}) \subset \mathcal{F}$ and $G_{\mathcal{D}}(\mathcal{F}) \subset \mathcal{J}$.

Let W be a finite-dimensional complex of S-modules. Then $G(W)$ is a finite-dimensional complex of free Λ-modules. Hence, $G_{\mathcal{D}}(F) \subset \mathcal{J}$. Conversely, let V be a complex of free Λ-modules. The module V has a finite filtration, its factors having one generator. For complexes V' with one generator, the space $H^*(F_{\mathcal{D}}(V'))$ is one-dimensional. Since $F_{\mathcal{D}}$ is faithful functor, we have $\dim H^*(F_{\mathcal{D}}(V')) < \infty$. By Lemma $F_{\mathcal{D}}(V) \in \mathcal{F}$, i.e., $F_{\mathcal{D}}(\mathcal{J}) \subset \mathcal{F}$. The proposition is proved. □

Recall (Proposition 3.8) that $\mathcal{F}$ is a thick subcategory of $\mathcal{D}^b(S)$.

6.15. Corollary. $\mathcal{J}$ *is a thick subcategory of* $\mathcal{D}^b(\Lambda)$ *and categories* $\mathcal{D}^b(S)/\mathcal{F}$ *and* $\mathcal{D}^b(\Lambda)/\mathcal{J}$ *are equivalent.*

7. The category $\mathcal{A}$; equivalence of categories

7.1. One of the aims of this section is to prove the statement c) from sec. 3.7 on the relation of $\mathcal{M}^0(\Lambda)$ and $\mathcal{D}^b(S)/\mathcal{F}$. For this, it is convenient to introduce one very interesting category of complexes of Λ-modules; i.e., the category $\mathcal{A}$. It also helps to make a very natural relation between our description of sheaves on $\mathbb{P}$ and that by A.A. Beilinson [B] (see 6.13-6.16).

Recall that $\mathcal{J}$ stands for a full subcategory of $\mathcal{D}^b(\Lambda)$ consisting of objects isomorphic to finite complexes of finite-dimensional free Λ-modules (see 5.14). The Proposition in 5.14, the first part of Proposition 3.8 and Theorem 5.7 easily imply the following lemma.

Lemma. (i) $\mathcal{J}$ *is a thick subcategory of the triangulated category* $\mathcal{D}^b(\Lambda)$ *so we may consider the quotient category* $\mathcal{D}^b(\Lambda)/\mathcal{J}$*.*

(ii) *The functors* $F\colon \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{D}^b(S)$ *and* $G_{\mathcal{D}}\colon \mathcal{D}^b(S) \longrightarrow \mathcal{D}^b(\Lambda)$ *extend to equivalence of categories* $\mathcal{D}^b(S)/\mathcal{J} \simeq \mathcal{D}^b(\Lambda)/\mathcal{F}$

7.2. The main result of this section is the following theorem.

Theorem. *The categories* $\mathcal{D}^b(\Lambda)/\mathcal{F}$ *and* $\mathcal{M}^b(\Lambda)/\mathcal{P}$ *are equivalent.*

To prove this theorem we introduce one more triangulated category, $\mathcal{A}$, and construct functors

$$\mathcal{D}^b(\Lambda) \xrightarrow{\alpha} \mathcal{A} \xleftarrow{\delta} \mathcal{M}^0(\ Lambda) \tag{93}$$

It turns out that the functor δ defines equivalence of categories. The functor α has the kernel, which coincides exactly with $\mathcal{J}$, so α defines equivalence of $\mathcal{D}^b(\Lambda)/J$ and $\mathcal{A}$.

7.3. The category $\mathcal{A}$. Let the objects of $\mathcal{A}$ be the complexes $A = \{A_i, i \in Z\}$ satisfying the following conditions:

a) Each A_i is a free module with a finite number of generators, i.e., $A_i = \overset{n_i}{\underset{l=1}{\oplus}} \Lambda[r_{i,l}]$.

b) The complex A is acyclic, i.e., $H^i(A) = 0$ for any i. As in Section 5 it is convenient to assume that the differentials $\partial_i : A_i \longrightarrow A_{i+1}$ supercommute with the Λ-action, i.e., $\partial_i \xi = -\xi \partial_i$. The complex morphisms up to homotopy equivalence are *morphisms* of A. The triangles isomorphic to

$$A \xrightarrow{u} B \longrightarrow C_u \longrightarrow T(A) \tag{94}$$

where $u\colon A \longrightarrow B$ is an arbitrary morphism, C_u is the cone of u (see Appendix) are fixed.

7.4. The category it is non-trivial since we consider complexes A infinite both ways. More precisely, the following simple lemma takes place.

Lemma. *Let the complex $A \in \mathcal{A}$ be bounded from one side. Then A is isomorphic to zero in $\mathcal{A}$.*

Proof. Let, for example, $A_i = 0$ for $i \leq 0$. Then $0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots$ is an injective resolution of the zero A-module 0. Since any two injective resolutions are homotopic equivalent, A is isomorphic to 0 in $\mathcal{A}$. □

7.5. A more general statement is also valid, however, we shall not need it.

Proposition. *Suppose $A = \{A_i\}$ and $B = \{B_i\}$ are two objects of $\mathcal{A}$ and the morphisms $f_i\colon A_i \longrightarrow B_i$ commuting with differentials are defined for $i > i_0$. Then there is the unique morphism $f\colon A \longrightarrow B$ in $\mathcal{A}$ continuing $\{f_i\}$. If f_i are isomorphisms then so is f, if $\{f_i\}$ and $\{g_i\}$ are two homotopic sets of morphisms, where $i > i_0$, then their continuations f and g are equal in $\mathcal{A}$.*

This proposition shows that $\mathcal{A}$ may be considered as the category of "right tails" $\{A_i, i \geq i_0\}$, i.e., exact sequences

$$A_{i_0} \longrightarrow A_{i_0+1} \longrightarrow \dots \tag{95}$$

of free finite-dimensional Λ-modules. The similar statement is valid, surely, for the "left tails" $\{A_i \mid i \leq i_0\}$

7.6. Let us construct the special complex $\Delta \in \mathcal{A}$ needed in the sequel.

The right-hand side of this complex is the (shifted) Koszul complex (see [Ha2], §III.3, [GH], ch.5, §3). Namely, let as above $X = \Theta^*$ be the space dual to X and $S^l(X)$ be the l-th symmetric power of X. For $l \geq 0$, set

$$\Delta_l = \Lambda[-n-1] \otimes S^l(X) \tag{96}$$

Δ_l is, surely, free Λ- module with C^l_{n+l} generators (each of degree $l-n-1$). Define the operator

$$\partial_l : \Delta_l \longrightarrow \Delta_{l+1} \tag{97}$$

setting

$$\partial_l(\lambda \otimes s) = \sum \xi_i \lambda \otimes x_i s \tag{98}$$

where $\{\xi_i\}$ and $\{x_i\}$ are dual bases in Θ and X, respectively.

The left-hand side of the complex Δ is dual to the right-hand side. Namely, define Δ_{-l} for $l > 0$ by the formula

$$\Delta_{-l} = (\Delta_{-l})^* \tag{99}$$

Let $\partial_{-l} = -\partial^*_{l-2}$. Thus, all modules Δ_l and all operators ∂_l, but ∂_0, are defined. To glue the left-hand side and the right-hand side of the complex Θ note that

$$\Delta_0 = \Lambda[-n-1]\Delta_{-l} = (\Lambda[-n-1])^* = \Lambda. \tag{100}$$

Let us define the operator $\partial_{-1}\colon \Lambda \longrightarrow \Lambda[-n-1]$ setting

$$\partial_{-1}(1) = \xi_0 \dots \xi_n e \tag{101}$$

($1 \in \Lambda$ and e is a generator of $\Lambda[-n-1]$. Note that $\Delta = F \circ G(1)$, where F and G are functors of section 5 1 is the trivial one-dimensional complex of Λ-modules.

Lemma. (i) $\Delta \in \mathcal{A}$

(ii) *In Δ we have* $\operatorname{Ker} \partial_0 = \operatorname{Im} \partial_{-1} = 1$ *is the one-dimensional module with the generator of degree* 0.

Proof. (i) It is clear that Δ satisfies the condition a) in the definition of objects of $\mathcal{A}$. The exactness in terms Δ_l, where $l \geq 1$, follows from the exactness of the Koszul complex (see [6] ch.5 section 3). The exactness in terms Δ_l, where $l \leq -2$ follows from the duality. The exactness in terms Δ_0 and Δ_{-1} is most easily to verify straightforwardly.

(ii) This is evident.

Note that

$$\Delta^+ = \{0 \longrightarrow 1 \longrightarrow \Delta_0 \longrightarrow \Delta_{-1} \longrightarrow \dots\} \tag{102}$$

and

$$\Delta^- = \{\dots \longrightarrow \Delta_{-2} \longrightarrow \Delta_{-1} \longrightarrow 1 \longrightarrow \dots\} \tag{103}$$

are injective and projective resolutions of the trivial one-dimensional module 1 (in $\mathcal{M}^b(\Lambda)$), respectively, and Δ is the cone of the composition $\Delta^- \longrightarrow 1 \longrightarrow \Delta^+$ □

7.7. The construction of the functor $\alpha\colon \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{A}$. For any complex $X \in C^b(\Lambda)$, set $\alpha(X) = X \otimes \Delta$. It is clear that $\alpha(X)$ is an object of $\mathcal{A}$.

Let $\varphi\colon X \longrightarrow Y$ be a complex morphism. Set $\alpha(\varphi) = \varphi \otimes 1\colon \alpha(X) \longrightarrow \alpha(Y)$. Let us prove that α defines a functor from $\mathcal{D}^b(A)$ to $\mathcal{A}$.

It is clear that if φ is homotopic to ψ, then $\alpha(\varphi)$ is homotopic to $\alpha(\psi)$. Therefore it suffices to show that if $\varphi\colon X \longrightarrow Y$ is a quasiisomorphism, then $\alpha(\varphi)\colon \alpha(X) \longrightarrow \alpha(Y)$ is a homotopic equivalence. In the following lemma a more general result is contained; we will need it in what follows.

A complex morphism $i\colon X \longrightarrow Y^+$, where Y^+ is a bounded from the left complex consisting of free finite-dimensional Λ-modules, and i is quasiisomorphism, is said to be an *injective resolution of the complex* $X \in C^b(\Lambda)$.

A *projective resolution* $j : Y^- \longrightarrow X$ is similarly defined.

Lemma. *Let $X \longrightarrow Y^+$ and $Y^- \longrightarrow X$ be respectively an injective and a projective resolutions of $X \in C^b(\Lambda)$, let C_u be the cone of the composition $Y^- \longrightarrow X \longrightarrow Y^+$. Then*

(i) $C_u \in A$

(ii) *If X' is quasiisomorphic to X and $C_{u'}$ is recovered from X' in the same way as C_u is recovered from X, then C_u and $C_{u'}$ are isomorphic in $\mathcal{A}$. In particular, $C_u \sim \alpha(X)$ in $\mathcal{A}$.*

Proof. To establish (i) we must verify conditions a)- b) in 6.3. Since in $\mathcal{M}_b(\Lambda)$ the injective and projective objects coincide, a) is obvious. Further, b) holds since the cone of the quasiisomorphism u is a cyclic. The heading (ii) holds since any two projective (injective) resolutions of quasiisomorphic objects are homotopic equivalent. (It follows from Proposition 7.10 (Appendix), see also [Ha1], sec. 1.4).

Since

$$X \otimes \Delta^- \longrightarrow X, \quad X \longrightarrow X \otimes \Delta^+ \tag{104}$$

are a projective and an injective resolutions of X (see 6.6) and $\alpha(X)$ is the cone of the composition $X \otimes \Delta^- \longrightarrow X \longrightarrow X \otimes \Delta^+$, we obtain the functor $\alpha\colon \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{A}$. □

7.8. Let us now prove that *the kernel of α coincides with the subcategory $\mathcal{J} \subset \mathcal{D}^b(\Lambda)$.*

Proposition. *The object $\alpha(x) \in \mathcal{A}$ is isomorphic to* 0 *in $\mathcal{A}$ if and only if $X \in \mathcal{J}$.*

Proof. 1) Let $X \in \mathcal{J}$. Let us prove that $\alpha(X)$ is isomorphic to 0 in $\mathcal{A}$. We may assume that a projective and an injective resolutions of X coincide with X. By Lemma 6.7 $\alpha(X)$ is isomorphic in $\mathcal{A}$ to the bounded complex (the cone of identity mapping of X in X) so, by Lemma 6.4 $\alpha(X) \sim 0$.

2) Let $\alpha(X) \sim X$. Let us prove that $X \in \mathcal{J}$. Let $Y^- \longrightarrow X$ and $X \longrightarrow Y^+$ are projective and injective resolutions of X (e.g. resolutions (6.1)). By Lemma 6.7 we may replace $\alpha(X)$, by the cone C of the through mapping $Y^- \longrightarrow X \longrightarrow Y^+$ and assume that there is the homotopy $\{k_i\colon C_i \longrightarrow Ci-1$ of the identity and zero morphisms in C, i.e.,

$$\partial_{i-1}k_i + k_{i+1}\partial_i = 1. \tag{105}$$

Let i be such that $X_j = 0$ and $Y_j^- = 0$ for $j \geq i$. Set $U = \operatorname{Im} \partial_{i+1}$. Then, first, the complex

$$\tau_{i+1}Y^+ = \{\dots \longrightarrow Y_i^+ \longrightarrow Y_{i+1}^+ \longrightarrow U \longrightarrow 0\} \tag{106}$$

is quasiisomorphic to Y, hence, is quasiisomorphic to X. On the other hand, since $C_j = Y^+_{j+1}$ for $j \geq i$, (6.2) implies that morphisms

(107) $$k_i \colon Y^+_{j+1} \longrightarrow Y^+_i \text{ and } k_{i+1} \colon Y^+_{i+2} \longrightarrow Y^+_{i+1}$$

satisfy

(108) $$\partial_i k_i + k_{i+1}\partial_{i+1} = 1$$

on Y^+_{i+1}. This implies that U is the direct summand of Y^+_{i+1} (with the complementary $\operatorname{Im}\partial_i k_i$). Since $Y^+_{i+1} \in \mathcal{P}$, we have $U \in \mathcal{P}$ so that $\tau_{i+1}Y^+ \in \mathcal{J}$ and $X \in \mathcal{J}$. Proposition is proved. □

7.9. Lemma. *Let $\zeta \colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be an exact functor of a triangulated categories and $\mathcal{D} \in \mathcal{C}_1$ a full subcategory consisting of all X such that $\zeta(X)$ is isomorphic to zero. Then $\mathcal{D}$ is a thick subcategory of $\mathcal{C}_1$ and ζ defines a functor $\mathcal{C}_1/\mathcal{D} \longrightarrow \mathcal{C}_2$.*

Proof easily follows from the definition of a thick subcategory. Details are left to the reader.

Proposition 6.8 and Lemma 6.9 imply that α defines a functor

(109) $$\alpha' \colon \mathcal{D}^b(\Lambda)/\mathcal{J} \longrightarrow \mathcal{A}.$$

7.10. We will prove that α' is equivalence of categories. This may be done in several ways and the one presented here is to construct the inverse of α' explicitly.

For each pair of integers $i \leq j$, we construct a functor $\beta_{i,j} \colon \mathcal{A} \longrightarrow \mathcal{D}^b(\Lambda)/cJ$ as follows. Let

(110) $$A = \{\ldots \longrightarrow A_{-1} \longrightarrow A_0 \longrightarrow A_1 \longrightarrow \ldots\} \in \mathcal{A}.$$

Consider the complex

(111) $$\{0 \longrightarrow A_i \longrightarrow \ldots \longrightarrow A_{j-1} \longrightarrow B \longrightarrow 0\},$$

where $B = \operatorname{Ker}(\partial_j \colon A_j \longrightarrow A_{j+1}) = \operatorname{Im}(\partial_{j-1} \colon A_{j-1} \longrightarrow A_j)$. Denote by $\beta_{i,j}(A)$ the image of this complex in $D^b(\Lambda)/\mathcal{J}$.

Lemma. (i) *$\beta_{i,j}$ defines a functor from A into $\mathcal{D}^b(\Lambda)/\mathcal{J}$.*

(ii) *For distinct i, j, the functors $\beta_{i,j}$ are isomorphic.*

(iii) *$\beta_{i,j}$ is a faithful functor of triangulated categories.*

Proof. To verify (i), it suffices to show that if $A \in \mathcal{A}$ is homotopic to zero, then the complex (6.3) belongs to $\mathcal{J}$, Let $\{k_l \colon A_l \longrightarrow A_{l-1}\}$ be the homotopy of the identity and zero mappings. As in the proof of Proposition 6.8 it is easy to show that B in the complex (6.3) is the direct summand of A, (with the complementary $\operatorname{Im} k_{j+1}\partial_j$). Hence $B \in \mathcal{P}$ and (6.3) $\in \mathcal{J}$.

Let us prove (ii). It suffices to verify that $\beta_{i,j} \sim \beta_{i+1,j}$ for $i < j$ and $\beta_{i,j} \sim \beta_{i,j+1}$ for $i < j$. To construct an isomorphism $\beta_{i,j} \sim \beta_{i+1,j}$, we need to fix an isomorphism $\theta_A \in \operatorname{Hom}_{\mathcal{D}^b(\Lambda)/\mathcal{J}}(\beta_{i,j}(A), \beta_{i+1,j}(A))$ for any $A \in \mathcal{A}$ such that the diagram

(112) $$\begin{array}{ccc} \beta_{i,j}(A) & \xrightarrow[\theta_A]{} & \beta_{i+1,j}(A) \\ \big\downarrow{\scriptstyle \beta_{i,j}(\varphi)} & & \big\downarrow{\scriptstyle \beta_{i+1,j}(\varphi)} \\ \beta_{i,j}(A') & \xrightarrow[\theta'_A]{} & \beta_{i+1,j}(A') \end{array}$$

is commutative for any $\varphi \in \operatorname{Hom}_{\mathcal{A}}(A, A')$. It is easy to see that as θ_A we may pick the natural morphism

(113) $$\{A_i \longrightarrow A_{i+1} \longrightarrow \ldots \longrightarrow A_{j-1} \longrightarrow B\} \longrightarrow \{A_{i+1} \longrightarrow \ldots A_{j-1} \longrightarrow B\},$$

which is isomorphism in $\mathcal{D}^b(\Lambda)/J$ (since A_i is free). The isomorphism $\beta_{i,j} = \beta_{i,j+1}$ is done similarly.

Now it suffices to prove the heading (iii) for some morphism $\beta_{i,j}$, e.g., for $\beta_{0,0}$. Let us prove, for example, that if $\varphi \colon A \longrightarrow A'$ is homotopic to zero, then $\beta_{0,0}(\varphi) = 0$ in $\mathcal{D}^b(\Lambda)/J$. Let $k_i \colon A_i \longrightarrow A'_{i-1}$ be a homotopy, i.e.,

(114) $$k_{i+1}\partial_i + \partial'_{i-1}k_i = \varphi \colon A_i \longrightarrow A'_{i-1}.$$

On $B = \operatorname{Ker}(\partial_0 \colon A_0 \longrightarrow A_1)$, we have $\varphi_0 = \partial'_{-1}k_0$. Hence, the morphism

(115) $$\beta_{0,0}(\varphi) \colon \{\ldots \longrightarrow 0 \longrightarrow B_0 \longrightarrow 0 \longrightarrow \ldots\} \longrightarrow \{\ldots \longrightarrow 0 \longrightarrow B' \longrightarrow 0 \longrightarrow \ldots\}$$

factorizes through the complex $\{\ldots \longrightarrow 0 \longrightarrow A'_{-1} \longrightarrow 0 \longrightarrow \ldots\} \in \mathcal{J}$, i.e., $\beta_{0,0}(\varphi) = 0$ in $\mathcal{D}^b(\Lambda)/\mathcal{J}$.

The other statements of (iii) are proved similarly. Lemma is proved. □

7.11. Proposition. *The functors $\beta_{i,j}\circ\alpha$ and $\alpha\circ\beta_{i,j}$ are isomorphic to the identity functors in $\mathcal{D}^b(\Lambda)/\mathcal{J}$ and $\mathcal{A}$ respectively.*

Proof. Let $A\in\mathcal{A}$ and

$$B=\operatorname{Ker}(\partial_0\colon A_0\longrightarrow A_1)=\operatorname{Im}(\partial_{-1}\colon A_{-1}\longrightarrow A_0). \tag{116}$$

Then

$$\ldots\longrightarrow A_{-2}\longrightarrow A_{-1}=Y^- \text{ and } A_0\longrightarrow A_1\longrightarrow\cdots=Y^+ \tag{117}$$

are projective and, respectively, injective resolutions of the complex

$$\beta_{0,0}(A)=\{\ldots\longrightarrow 0\longrightarrow B\longrightarrow\ldots, \tag{118}$$

and A the cone of the composition $Y^-\longrightarrow\beta_{0,0}(A)\longrightarrow Y^+$.

Therefore by Lemma 6.7 $\alpha\circ\beta_{0,0}(A)$ is isomorphic to A in $\mathcal{A}$. It is easy to show that this isomorphism is a functor in A, i.e., $\alpha\circ\beta_{0,0}(A)\sim\mathrm{id}_{\mathcal{A}}$. By Lemma 6.1 $\alpha\circ\beta_{i,j}(A)\sim\mathrm{id}_{\mathcal{A}}$ for any $i\le j$.

Conversely, let $X\in\mathcal{D}^b(\Lambda)/\mathcal{J}$ and $X=\{X_k\longrightarrow\ldots\longrightarrow X_l\}$. Let

$$Y^-=X\otimes\Delta^- \text{ and } Y^+=X\otimes\Delta^+ \tag{119}$$

be two resolutions of X so that $\alpha(X)$ is the cone of the composition $Y^-\longrightarrow XY^+$. Fix $i<k$ and $j>l$ and consider complexes

$$\{Y_i^-\longrightarrow\ldots\longrightarrow Y_l^-\}=\tilde{Y}^- \tag{120}$$

and

$$\{Y_k^+\longrightarrow\ldots\longrightarrow Y_{j-1}^+\longrightarrow B\}=\tilde{Y}^+ \tag{121}$$

where

$$B=\operatorname{Im}(Y_{j-1}^+\longrightarrow Y_j^+) \tag{122}$$

Then as earlier, we have morphisms

$$\tilde{Y}^-\longrightarrow X,\quad X\longrightarrow\tilde{Y}^+ \tag{123}$$

And $\beta_{i,j}$ is, clearly, the cone of the through mapping $\tilde{Y}^-\longrightarrow X\longrightarrow\tilde{Y}^+$ so that in $D^b(\Lambda)/J$ there is the fixed triangle

$$\tilde{Y}^-\longrightarrow\tilde{Y}^+\longrightarrow\beta_{i,j}(\alpha(X))\longrightarrow T(\tilde{Y}^-) \tag{124}$$

Since $\tilde{Y}^-\in J$, the objects $\tilde{Y}^+$ and $\beta_{i,j}(\alpha(X))$ are isomorphic in $D^b(\Lambda)/\mathcal{J}$. Further, it is clear that (since $j>l$) $X\longrightarrow Y^+$ is quasiisomorphism. Hence $\tilde{Y}^+$ is isomorphic to X in $D^b(\Lambda)$ and in $D^b(\Lambda)/\mathcal{J}$. Hence X is isomorphic to $\beta_{i,j}(\alpha(X))$ in $D^b(\Lambda)/\mathcal{J}$. It is easy to show that the constructed isomorphism $X\longrightarrow\beta_{i,j}(\alpha(X))$ is the functor in X. Therefore Proposition 6.11 is proved. □

7.12. Proposition 6.11 makes categories $D^b(\Lambda)/\mathcal{J}$ and $\mathcal{A}$ equivalent.

Similarly (but much more simpler) the equivalence of categories $\mathcal{A}$ and $\mathcal{M}^0(\Lambda)$ is proved. We will confine ourselves to corresponding functors leaving all the details to the reader.

The functor $\gamma\colon\mathcal{A}\longrightarrow\mathcal{M}^0(\Lambda)=\mathcal{M}^b(\Lambda)/\mathcal{P}$ assigns to the complex $A=\{A_i\}$ the module $M=\operatorname{Ker}(\partial_0\colon A_0\longrightarrow A_1)$. As in the proof of Lemma 6.10 (i) it is easy to show that if a morphism $\varphi\colon A\longrightarrow A'$ is homotopic to 0, then $\gamma(\varphi)\colon\gamma(A)\longrightarrow\gamma(A')$ factorizes through a free module (namely, A'_{-1}).

The functor $\delta\colon\mathcal{M}^0(\Lambda)\longrightarrow\mathcal{A}$ is defined by the formula

$$\delta(V)=V\otimes\Delta, \tag{125}$$

where Δ is the complex defined in 6.6. Lemma 6.4 implies that if $V\in\mathcal{P}$, then $\delta(V)$ is isomorphic to 0 in $\mathcal{A}$.

7.13. In conclusion of this section we account a relation of the above result with a result by A.A. Beilinson [B], who gave a somewhat different description of the derived category $\mathcal{D}^b(\mathsf{Sh})$.

Recall that $\Lambda[i]$ is a free graded Λ-module with one generator of degree i. Denote by $\mathcal{M}_{[0,n]}(\Lambda)$ the full subcategory of $\mathcal{M}^b(\Lambda)$ consisting of finite direct sums of modules $\Lambda[i]$, where $0 \le i \le n$. Let $C_{[0,n]}(\Lambda)$ be the full subcategory of $C^b(\Lambda)$, consisting of finite complexes of modules of $\mathcal{M}_{[0,n]}(\Lambda)$ and $\mathcal{K}_{[0,n]}(\Lambda)$ the corresponding homotopy category.

Similarly, let $S[i]$ be the free graded S-module with one generator of degree i and $C_{[0,n]}(S)$ the full subcategory of $\mathcal{M}_{[0,n]}(S)$ consisting of finite direct sums of modules $S[i]$ where $0 \le i \le n$. Define the homotopy category $\mathcal{K}_{[0,n]}(S)$ consisting of finite complexes of S-modules; similarly define $\mathcal{K}_{[0,n]}(\Lambda)$.

The categories $\mathcal{K}_{[0,n]}(\Lambda)$ and $\mathcal{K}_{[0,n]}(S)$ are full triangulated subcategories of $\mathcal{D}^b(\Lambda)$ and $\mathcal{D}^b(S)$, respectively. The main theorem of Beilinson's paper [B] claims that each of categories $K_{[0,n]}(\Lambda)$ and $K_{[0,n]}(S)$ is equivalent to the category $D^b(\mathsf{Sh})$ as triangulated categories.

By Theorem 4.5 and 6.3–6.12 the category $\mathcal{A}$ introduced in 6.3 is equivalent to the category $D^b(\mathsf{Sh})$. Let us replace $\mathcal{A}$ by its full subcategory $\mathcal{A}'$ consisting of complexes $\{\ldots \longrightarrow A_{-1} \longrightarrow A_0 \longrightarrow A_1 \longrightarrow \ldots\}$ satisfying a) and b) from 6.3 and also

c) For each r, the equality $r_{i,l} = r$ (see 6.3, a)) is only satisfied by a finite number of pairs (i,l).

It is easy to verify that everywhere in 6.3-6.12 $\mathcal{A}$ may be replaced by $\mathcal{A}'$ (since the complex Δ from 6.6 belongs to $\mathcal{A}'$). In fact, the embedding functor $\mathcal{A}' \longrightarrow \mathcal{A}$ defines the equivalence of $\mathcal{A}$ and $\mathcal{A}'$.

Let us construct functors

$$\gamma_\Lambda \colon \mathcal{A}' \longrightarrow \mathcal{K}_{[0,n]}(\Lambda), \quad \gamma_S \colon \mathcal{A}' \longrightarrow \mathcal{K}_{[0,n]}(S), \tag{126}$$

which are equivalences of the corresponding categories. We skip proof of equivalence. It is similar to the above arguments.

7.14. In each Λ-module $V \in \mathcal{M}^b(\Lambda)$, let us introduce two filtrations by Λ-submodules, $V\{l\}$ and $V(l)$, as follows. Let $V = \oplus V_j$ be the decomposition in to homogeneous components. Set

$$V(l) = \Lambda(\oplus_{j \ge l} V_j), \quad V\{l\} = \Lambda(\oplus_{j \le l} V_j). \tag{127}$$

The filtration $V\{l\}$ is increasing, while $V(l)$ is decreasing. For any morphism $\varphi \colon V \longrightarrow W$ in $\mathcal{M}^b(\Lambda)$, we have

$$\varphi(V\{l\}) \subseteq W\{l\}, \quad \varphi(V(l)) \subseteq W(l). \tag{128}$$

Besides

$$V\{l_1\} \cap V(l_2) = \{0\} \tag{129}$$

for $l_2 - l_1 > n+1$.

For any $V \in \mathcal{M}^b(\Lambda)$, set

$$\begin{aligned} \eta(V) &= V\{n\}/V\{-1\}, \\ \eta'(V) &= V\{-1\} \cap V(0). \end{aligned} \tag{130}$$

It is clear that η and η' define two functors from $\mathcal{M}^b(\Lambda)$ into itself. The modules $\eta(V)$ and $\eta'(V)$ satisfy the following easily verified conditions:

a) If V is free then $\eta(V) \in \mathcal{M}_{[0,n]}(\Lambda)$ i.e., $\eta(V)$ is the free module with generators of degree between 0 and n.

b) For any $V \in \mathcal{M}^b(\Lambda)$, the module $\eta'(V)$ has non-zero homogeneous elements only in degrees between 0 and n (it easily follows from (6.4)).

7.15. The construction of the functor γ_Λ. Let

$$A = \{\ldots \longrightarrow A_{-1} \longrightarrow A_0 \longrightarrow A_1 \longrightarrow \ldots\} \in \mathcal{A}. \tag{131}$$

Denote by $\gamma_\Lambda(A)$ the complex

$$\gamma_\Lambda(A) = \{\ldots \longrightarrow \eta(A_{-1}) \longrightarrow \eta(A_0) \longrightarrow \eta(A_1) \longrightarrow \ldots\}. \tag{132}$$

The property c) of the category A (see 6.13) and the property a) above easily imply that

$$\gamma_\Lambda(A) \in C_{[0,n]}(\Lambda). \tag{133}$$

The action of γ_Λ on complex morphisms is naturally defined and it is clear that γ_Λ preserves the homotopy equivalence of morphisms. Therefore γ_Λ defines a functor (denoted by the same letter)

$$\gamma_\Lambda \colon \mathcal{A} \longrightarrow \mathcal{K}_{[0,n]}(\Lambda). \tag{134}$$

Proposition. *$\gamma_\Lambda\colon \mathcal{A} \longrightarrow \mathcal{K}_{[0,n]}(\Lambda)$ is equivalence of triangulated categories.*

7.16. Construction of the functor γ_S. The functor γ_S is more difficult to construct. First, denote by $\tilde{\mathcal{D}}^b(\Lambda)$ the full subcategory of $\mathcal{D}^b(\Lambda)$ consisting of complexes X such that $H^*(X)$ consists of homogeneous elements in degrees j, where $0 \leq j \leq n$.

Proposition. *The functor $F_{\mathcal{D}}\colon \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{D}^b(S)$ identifies $\mathcal{D}^b(\Lambda)$ with the complete subcategory $\mathcal{K}_{[0,n]}(S)$ of $\mathcal{D}^b(S)$.*

Further, from the functor η' we recover (similarly to γ_Λ) the functor $A \longrightarrow \mathcal{K}^b(\Lambda)$. Let $\gamma''\colon A \longrightarrow \mathcal{D}^b(\Lambda)$ be the composition of this functor with the natural functor $\mathcal{K}^b(\Lambda) \longrightarrow \mathcal{D}^b(\Lambda)$ The property 6.14 b) implies $\gamma''(A) \in D^b(\Lambda)$ for any $A \in A$.

Proposition. *γ'' makes the equivalence of A and $\tilde{\mathcal{D}}^b(\Lambda)$.*

Now set

(135) $$\gamma_S = F_{\mathcal{D}} \circ \gamma''.$$

Corollary. *γ_S makes equivalence of $\mathcal{A}$ and $\mathcal{K}_{[0,n]}(S)$.*

8. Appendix. The derived categories

8.1. To the authors' knowledge, definitions and results on derived categories are described in two places: in the first chapter of the book by R. Hartshorne [Ha1] and in Verdier's talk [V]. Following these two expositions we give a comparatively short list of basic definitions and facts used in this article.

8.2. An additive category C endowed with

a) a functor $T\colon C \longrightarrow C$ which is an isomorphism;

b) sextuples (X, Y, Z, u, v, w), where $u\colon X \longrightarrow Y$, $v\colon Y \longrightarrow Z$, $w\colon Z \longrightarrow T(X)$ is called a *triangulated category.*

The morphism T is called the *translation functor* and the fixed sextuples are called *triangles in C*. We will usually express each triangle in the form

(136) $$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

A commutative diagram

(137) $$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ \varphi\downarrow & & \psi\downarrow & & \theta\downarrow & & T(\varphi)\downarrow \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow{w'} & T(X') \end{array}$$

is called a *triangle morphism* provided the following axioms are satisfied:

(TR1) Each set (X', Y', Z', u', v', w') isomorphic to a triangle (i.e., in the diagram (A.1), φ, ψ, θ are isomorphisms) is a triangle. Each morphism $u\colon X \longrightarrow Y$ enters a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$. The set $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a triangle for any X.

(TR2) $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ and $Y \xrightarrow{v} Z \xrightarrow{w} T(X)X \xrightarrow{-T(u)} T(Y)$ are or are not triangles simultaneously.

(TR3) Given triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') and morphisms $\varphi\colon X \longrightarrow X'$ and $\psi\colon Y \longrightarrow Y'$ commuting with u and u', respectively there is a morphism $\theta\colon Z \longrightarrow Z'$ (not necessarily unique) such that (φ, ψ, θ) is a triangle morphism.

(TR4) Given triangles

(138) $$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{\alpha} & Z' & \xrightarrow{\beta} & T(X) \\ Y & \xrightarrow{v} & Z & \xrightarrow{\gamma} & X' & \xrightarrow{\delta} & T(Y) \\ X & \xrightarrow{vu} & Z & \xrightarrow{\varepsilon} & Y' & \xrightarrow{\eta} & T(X) \end{array}$$

there exist $'\varphi\colon Z \longrightarrow Z'$ and $\psi\colon Y' \longrightarrow X'$ such that

(139) $$Z \xrightarrow{\varphi} Y' \xrightarrow{\psi} X' \overset{T(\delta)\alpha}{\circ} T(Z')$$

is a triangle and

$$\gamma = \varepsilon\psi, \quad \beta = \eta\varphi. \tag{140}$$

The axiom TR4 is called the *octahedron axiom*, since objects and morphisms involved it is convenient to present as vertices and edges of an octahedron.

A functor $F : C \longrightarrow C'$ from one triangulated category to another is *faithful* (or a ∂ *functor*) if it is additive, commutes with translations and transforms triangles into triangles.

8.3. Proposition. (see [Ha1], Proposition 1.1 (i), (iii)). *The composition of any two subsequent morphisms in a triangle equals* 0*. If, in the diagram (137), φ and ψ are isomorphisms, then so is θ.*

8.4. The most part of the known triangulated categories are given by the following construction.

Let $\mathcal{B}$ be an abelian category. A *complex* of objects of $\mathcal{B}$ is a set $X = \{X_i, d_i\}$ of objects of $\mathcal{B}$ and morphisms $d_i = d_{i,X} : X_i \longrightarrow X_{i+1}$ such that $d_{i+1}d_i = 0$. A *complex morphism* $f : X \longrightarrow Y$ is a set of morphisms $f_i : X_i \longrightarrow Y_i$ such that for any i we have

$$f_{i+1}d_{i,X} = d_{i,Y}f_i. \tag{141}$$

Denote by $C(\mathcal{B})$ the category of complexes of objects of B with the introduced morphisms. $C(\mathcal{B})$ is, naturally, abelian.

A morphism $f : X \longrightarrow Y$ is homotopic to 0 if there exists a set of morphisms $k_i : X_i \longrightarrow Y_{i-1}$ such that

$$f_i = d_{i-1,Y}k_i + k_{i+1}d_{i,X}. \tag{142}$$

Morphisms $f : X \longrightarrow Y$ homotopic to zero form, clearly, a subgroup $\mathrm{Ht}(X,Y) \subset \mathrm{Hom}_{C(\mathcal{B})}(X,Y)$,

Denote by $\mathcal{K}(\mathcal{B})$ the *homotopy category* of $\mathcal{B}$: its objects coincide with the objects of $C(\mathcal{B})$ (i.e., with complexes) and morphisms in $\mathcal{K}(\mathcal{B})$ are given by the formula

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{B})}(X,Y) = \mathrm{Hom}_{C(\mathcal{B})}(X,Y)/\mathrm{Ht}(X,Y) \tag{143}$$

Define the *cone Z of a morphism* $u \in \mathrm{Hom}_{C(\mathcal{B})}(X,Y)$ by setting

$$Z_i = X_{i+1} \oplus Y_i, \tag{144}$$

$$d_{i,Z}(x_{i+1}, y_i) = (-d_{i+1,X}(x_{i+1}), d_{i,Y}(y_i)) + u_{i+1}(x_{i+1})). \tag{145}$$

It is easy to verify that $Z \in C(\mathcal{B})$ and if we replace u with a homotopic morphism, then Z becomes replaced with an isomorphic object of $\mathcal{K}(\mathcal{B})$.

On $\mathcal{K}(\mathcal{B})$, define a triangulated category structure. Let X be an object of $\mathcal{K}(\mathcal{B})$. Set $(T(X))_i = X_{i+1}$, and $d_{i,T(X)} = -d_{i+1,X}$. Further, for any morphism $u : X \longrightarrow Y$ consider the set (X, Y, Z, u, v, w), where Z is the cone of U, while $v : Y \longrightarrow Z$ and $w : Z \longrightarrow T(X)$ are natural morphisms. Define a triangle in $\mathcal{K}(\mathcal{B})$ as any set isomorphic to a set obtained by such a construction for a morphism $u : X \longrightarrow Y$

Proposition. *$\mathcal{K}(\mathcal{B})$ endowed with the above functor T and a set of triangles is a triangulated category, i.e., axioms TR1-TR4 are verified.*

The proof of this proposition is sufficiently easy (see [Ha1], sec. 1.2).

8.5. A full triangulated subcategory $\mathcal{D}$ of a triangulated category $\mathcal{C}$ is said to be *thick* if it satisfies the following condition: let $u : X \longrightarrow Y$ be a morphism of $\mathcal{C}$, which factors through an object of $\mathcal{D}$ and is contained in a triangle (X, Y, Z, u, v, w), where $Z \in \mathcal{D}$ .Then $X, Y \in \mathcal{D}$.

Example. Let $Ac(\mathcal{B})$ be the full subcategory of $\mathcal{K}(\mathcal{B})$ consisting of complexes X such that $H^i(X) = 0$ for any i (such X are called acyclic). Then $Ac(\mathcal{B})$ is a thick subcategory in $\mathcal{K}(\mathcal{B})$.

8.6. The quotient category modulo a thick subcategory. Let $\mathcal{C}$ be a triangulated category and $\mathcal{D}$ the thick subcategory.

Theorem. *There are triangulated category $\mathcal{C}/\mathcal{D}$ and a faithful functor $Q : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{D}$ such that any faithful functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ (where $\mathcal{C}'$ is an arbitrary triangulated category) which transforms objects of $\mathcal{D}$ into zero objects of $\mathcal{C}'$ can be uniquely presented in the form $F = G \circ Q$, where $G : \mathcal{C}/\mathcal{D} \longrightarrow \mathcal{C}'$ is a faithful functor.*

We will not carry out the proof of this theorem in all details (see [Ha1], §1.3, 1.4). Let us say a few words on the construction of $\mathcal{C}/\mathcal{D}$.

Denote by $\mathcal{S} = \mathcal{S}_{\mathcal{D}}$ the set of morphisms $u : X \longrightarrow Y$ in $\mathcal{C}$ that may be inscribed in a triangle (X, Y, Z, u, v, w) where $Z \in \mathcal{D}$. The main idea in the construction of $\mathcal{C}/\mathcal{D}$ is that morphisms of $\mathcal{S}$ are now said to be invertible in the category $\mathcal{C}/\mathcal{D}$.

More precisely, the objects of $\mathcal{C}/\mathcal{D}$ coincide with the objects of $\mathcal{C}$. The set $\mathrm{Hom}_{\mathcal{C}/\mathcal{D}}(X, Y)$ is the set of diagrams

(146)
$$\begin{array}{ccc} Z & & \\ \downarrow{\scriptstyle s} & \searrow^{u} & \\ X & & Y \end{array}$$

where $s \in \mathcal{S}$ and the diagram (A.2) is identified with the diagram

(147)
$$\begin{array}{ccc} Z' & & \\ \downarrow{\scriptstyle s'} & \searrow^{u'} & \\ X & & Y \end{array}$$

if there is an object Z'' and morphisms $f\colon Z'' \longrightarrow Z$ and $f'\colon Z'' \longrightarrow Z'$ in $\mathcal{C}$ such that $uf = u'f'$ and $sf = s'f' \in \mathcal{S}$

To define the composition of morphisms

(148)
$$\begin{array}{ccc} U & & \\ \downarrow{\scriptstyle s} & \searrow^{u} & \\ X & & Y \end{array} \qquad \begin{array}{ccc} V & & \\ \downarrow{\scriptstyle t} & \searrow^{v} & \\ Y & & Z \end{array}$$

we must make the commutative diagram

(149)
$$\begin{array}{ccccc} W & & & & \\ \downarrow{\scriptstyle r} & \searrow^{w} & & & \\ U & & V & & \\ \downarrow{\scriptstyle s} & \searrow^{u} & \downarrow{\scriptstyle t} & \searrow^{v} & \\ X & & Y & & Z \end{array}$$

where $r \in \mathcal{S}$ and take as the composition

(150)
$$\begin{array}{ccc} W & & \\ \downarrow{\scriptstyle sr} & \searrow^{vw} & \\ X & & Z \end{array}$$

We may show that the diagram (A.3) always exists and different choices lead to equivalent diagrams(A.4). The other way to construct the quotient categories (accepted in [Ha1]) is to start directly from the set of morphisms $\mathcal{S}$ in $\mathcal{C}$ satisfying several conditions. (For details and proofs, see [Ha1], sec. 1.3.)

8.7. Let $\mathcal{B}$ be an abelian category, $Ac(\mathcal{B})$ the thick subcategory of the triangulated category $\mathcal{K}(\mathcal{B})$ consisting of acyclic objects. The triangulated category $\mathcal{D}(\mathcal{B}) = \mathcal{K}(\mathcal{B})/Ac(\mathcal{B})$ is called the *derived category* of $\mathcal{B}$.

Note that if two objects X and Y of $\mathcal{K}(\mathcal{B})$ are isomorphic in $\mathcal{D}(\mathcal{B})$, then $H^i(X) \simeq H^i(Y)$ for all i. But the converse statement is false in general.

8.8. Let $\mathcal{C}^+(\mathcal{B})$ be the full subcategory of $\mathcal{C}(\mathcal{B})$ consisting of bounded from the left complexes X (i.e., $X_i = 0$ for $i \ll 0$). The corresponding homotopy category $\mathcal{K}^+(\mathcal{B})$ is triangulated and the subcategory $Ac^+(\mathcal{B}) = Ac(\mathcal{B}) \cap \mathcal{K}^+(\mathcal{B})$ of a cyclic objects of $\mathcal{K}^+(\mathcal{B})$ is thick. Set $\mathcal{D}^+(\mathcal{B}) = \mathcal{K}^+(\mathcal{B})/Ac^+(\mathcal{B})$ The category $\mathcal{D}^-(\mathcal{B})$ of complexes bounded from the right and $\mathcal{D}^b(\mathcal{B})$ of complexes bounded from both sides are similarly defined. The functors

(151)
$$\begin{array}{ccccc} & & \mathcal{D}^+(\mathcal{B}) & & \\ & \nearrow & & \searrow & \\ \mathcal{D}^b(\mathcal{B}) & & & & \mathcal{D}(\mathcal{B}) \\ & \searrow & & \nearrow & \\ & & \mathcal{D}^-(\mathcal{B}) & & \end{array}$$

are naturally constructed and they are complete and fidel (i.e., they define embeddings of full subcategories). more general results are contained in [Ha1], sec. 1.4 and [V], sec. II.1).

Let us list several properties of derived categories.

8.9. Proposition. *Let $X, Y \in C^b(\mathcal{B})$, i.e., X and Y are bounded from both sides complexes of objects of B. Then $\mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(X,Y) = Hom_{\mathcal{K}(\mathcal{B})}(X,Y)$ in each of the following two cases:*
(i) *X consists of projective objects of $\mathcal{B}$;*
(ii) *X consists of injective objects of $\mathcal{B}$.*

8.10. Proposition ([Ha1], ch. 1, Prop. 4.7). *Let $\mathcal{B}$ be an abelian category with plenty of injective objects, $\mathcal{K}^+(J)$ the full subcategory of $\mathcal{K}^+(\mathcal{B})$ consisting of complexes such that all their components are injective. Then the natural functor $\mathcal{K}^+(J) \longrightarrow \mathcal{D}^+(\mathcal{B})$ defines the equivalence of categories.*

The similar statement holds for $\mathcal{D}^-(\mathcal{B})$ and projective objects.

9. Editor's remarks

9.1. When we endow Λ with the superalgebra structure, we have to endow with the superspace structure all the Λ-modules (unless we want to consider Λ as an algebra, not superalgebra). The modules considered in [B] are endowed with $\mathbb{Z}$-grading. Evidently, either the $\mathbb{Z}$-grading of a Λ-module M is *consistent*, i.e.

(152) $$M_{\bar 0} = \bigoplus_{i\equiv 1 \mod 2} M_i \text{ and } M_{\bar 1} = \bigoplus_{i\equiv 1 \mod 2} M_i,$$

or the $\mathbb{Z}$-grading of the Λ-module $\Pi(M)$ is consistent. The modules with consistent gradings correspond to bundles, whose fibers are even spaces. Classification of bundles over $\mathbb{P}^n$ with purely odd (i.e. $(0,k)$-dimensional) fibers is obtained from the results of [B], if we consider the modules M such that the $\mathbb{Z}$-grading of the module $\Pi(M)$ is consistent.

Suppose $\dim_{\mathbb{C}} V = (n+1, m)$ and $\mathcal{P}^{m,n}$ is the functor that to any commutative superalgebra C assigns the set of $(1,0)$-dimensional subsuperspaces in $V \otimes C$.

The bundles on $\mathcal{P}^{m,n}$ are described by finitely generated S- or E-modules, where $S = S(V)$ and $E = E(V)$, so that the passage from S to E is equivalent to fiber-wise application of the change of parity functor Π.

Suppose $\mathcal{F}$ is the full subcategory in $\mathcal{D}^b(S)$ ($\mathcal{D}^b(E)$, if $m \neq 0$ consisting of finite-dimensional S-modules (resp. E-modules).

Theorem. *The categories $\mathcal{D}^b(\mathsf{Coh})$, $\mathcal{D}^b(E)$ and $\mathcal{D}^b(S)$ are equivalent.*

Corollary. *The derived categories of coherent sheaves on $\mathcal{P}^{n,m}$ and $\mathcal{P}^{m,n}$, where $mn \neq 0$, are equivalent.*

9.2. The complex of integrable-differential forms. On the linear $(0,n)$-dimensional supermanifold $\mathcal{K}^{0,n}$ (the functions on $\mathcal{K}^{0,n}$ are the elements of $\Lambda(n)$ with generators $\xi = (\xi_1, \dots, \xi_n)$) define *differential forms* as elements of the commutative superalgebra that will be denoted usually by $\Omega(\mathcal{K}^{0,n})$ or just Ω with generators ξ and $\hat\xi$, where $p(\hat\xi_i) = \bar 0$ for $1 \leq 1 \leq n$. Setting $\deg \hat\xi_i = \deg \xi_i + 1 = 1$ for $1 \leq i \leq n$, we obtain a $\mathbb{Z}$-grading in Ω of the form $\Omega = \bigoplus_{i\geq 0} \Omega^i$. The operator $d = \sum \hat\xi_i \partial_i$ is called the *exterior differential.*

Suppose $\mathrm{Vol} = (\Omega^0)^*$. The space Vol is endowed with the structure of one-dimensional module over $\Lambda(n)$ with the distinguished generator 1^*.

Set $\Sigma_{-i} = (\Omega^i)^*$. In $\Sigma = \bigoplus_{i\geq 0} \Sigma_{-i}$, introduce the $\mathbb{Z}$-graded superspace structure setting $p(1^*) \equiv n \mod 2$ and $\deg 1^* = -n$.

It is convenient to denote elements of Σ_{-i} as Vol-valued polynomial functions in ξ_j and $\frac{\partial}{\partial\hat\xi_j}$ of degree i in $\frac{\partial}{\partial\hat\xi_j}$. They are called *integrable* i-forms. Setting

(153) $$\hat\xi_i\left(\frac{\partial}{\partial\hat\xi_j}\right) = \delta_{ij},$$

we define on Σ an Ω-module structure and the action of d. It is not difficult to verify that this action coincides with

(154) $$d^* \colon (\Omega^{i+1})^* = \Sigma_{-i-1} \longrightarrow \Sigma_{-i} = (\Omega^i)^*.$$

Berezin's integral is the mapping

(155) $$\int \colon \Sigma_0 \longrightarrow \mathbb{C}$$

dual to the embedding $\mathbb{C} \longrightarrow \Omega^0$.

Theorem (an analogue of the Poincaré lemma). *The sequence*

$$\dots \xrightarrow{d} \Sigma_{-i} \xrightarrow{d} \Sigma_0 \xrightarrow{\int} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \tag{156}$$

is exact.

The reader will easily see that the complex Δ described in [Be2] and the complex of integrable-differential forms is the same thing.

The complex of integrable-differential forms is the complex of $\mathfrak{vect}(0|n)$-modules. It turns out that for Lie superalgebras $\mathfrak{sl}(1|n)$, $\mathfrak{osp}(2|2n)$, $\mathfrak{svect}(0|n)$, $\mathfrak{h}(0|n)$ and $\mathfrak{pe}(n)$ acyclic complexes of free Λ-modules may also be constructed.

9.2.1. Problem. *For Lie superalgebras of type* $\mathfrak{sl}(n|m)$*, where* $1 \le m \le n$*, similar complexes (*m*-dimensional) may be constructed. What bundles correspond to these complexes?*

Generalize this question to complexes of modules over other vectorial Lie superalgebras.

9.3. Bundles on $\mathcal{P}^{1,1}$.

As follows from the above description of all bundles on $\mathcal{P}^{n,m}$ reduces to the hopeless linear algebra problem whenever n or m are greater than 1.

Let us sketch the explicit description in the single remaining case $n = m = 1$. Let us reduce the sheaf from $\mathcal{P}^{1,1}$ to $\mathbb{P}^1$ and decompose it into two parts: with the even fiber $\mathcal{E}$ and with the odd fiber $\mathcal{O}d$. Multiplying, if necessary, by $\mathcal{O}(n)$, where $n > 0$, we may assume that the sheaves $\mathcal{E}$ and $\mathcal{E}d$ are very ample. Suppose x, y, ξ are homogeneous coordinates on $\mathcal{P}^{1,1}$, where x and y are even and ξ is odd. The multiplication by ξ transforms $\mathcal{E}$ into $\mathcal{O}d$ and $\mathcal{O}d$ into $\mathcal{E}$. Since $\xi^2 = 0$, we obtain a complex.

Exercise. The homology of the operator ξ is 0.

Thus, we have two extensions

$$0 \longrightarrow \operatorname{Ker}\xi \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\operatorname{Ker}\xi \longrightarrow 0 \tag{157}$$

and

$$0 \longrightarrow (\operatorname{Ker}\xi)' \longrightarrow \mathcal{O}d \longrightarrow \mathcal{O}d/(\operatorname{Ker}\xi)' \longrightarrow 0 \tag{158}$$

Each extension is easy to describe, see ??[CE].

$$\begin{aligned} \operatorname{Ext}^1(\mathop{\oplus}_i \mathcal{O}(n_i), \mathop{\oplus}_j \mathcal{O}(m_j)) &= \mathop{\oplus}_{i,j} \operatorname{Ext}^1(\mathcal{O}(n_i), \mathcal{O}(m_j)) = \\ = \mathop{\oplus}_{i,j} \operatorname{Ext}^1(\mathcal{O}(n_i - m_j), \mathcal{O}) &= \mathop{\oplus}_{i,j} \begin{cases} 0 \text{ if } n_i - m_j \geq ??? \\ S^{m_j - n_i - 2}(V) \otimes \Lambda^2(V) \text{ if } n_i - m_j. \end{cases} \end{aligned} \tag{159}$$

Thus, from any four bundles on $\mathbb{P}^1$ we may recover a bundle on $\mathcal{P}^{1,1}$ depending on parameters that are described in the above formula.

9.3.1. Problem. *Why does* $H^1(\mathcal{O}(n))$ *coincide with the superspace of integrable* $-(n-2)$*-forms with constant coefficients while* $H^0(\mathcal{O}(n))$ *coincides with the superspace of integrable* n*-forms with constant coefficients on the* $(0, 2)$*-dimensional supermanifold?*

9.3.2. Problem. *It would be interesting to calculate dual derived categories of modules over superalgebras with quadratic relations that generalize symmetric and exterior algebras for other classical superdomains.*

9.3.3. Problem. *Describe bundles over classical superdomains of projective superspaces with non-degenerate bilinear forms a la* [Be2].

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