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Feliks Aleksandrovich Berezin (Obituary)

Feliks Aleksandrovich Berezin died on 14 July 1980 as the result of an accident.

Berezin's life came to an end in the flower of his mathematical talent—as testimony there remains not only what he had already succeeded in doing, but also his projects, his unfinished articles and books.

Berezin was born on 25 April 1931 in Moscow. After leaving school in 1948 he enrolled in the Faculty of Mechanics and Mathematics at the University of Moscow and graduated in 1953. For three years he taught mathematics in one of the Moscow schools, and then, from 1956 to the end of his life, he worked in the Faculty of Mechanics and Mathematics at the Moscow State University, first in a junior post and from 1962 on as a senior scientific worker attached to the chair of Function Theory and Functional Analysis.

The main pivot of Berezin's scientific interest was mathematical physics. Nowadays mathematical physics embraces many complicated and sometimes extremely abstract mathematical theories and constructions arising from the trend towards giving a clear meaning to fundamental physical theories (quantum physics, theory of gravitation, statistical physics, etc.). The fact that today there is a wide understanding of mathematical physics and that it is popular among mathematicians (and even among physicists) is due in no small measure to Berezin's role. More than twenty years ago he was enchanted by the beauty and significance of mathematical problems in quantum theory, and every since they have remained the leitmotiv of his mathematical creative work.

Berezin's first important paper dating from the middle of the 50's, before his conversion to mathematical physics, was that of 1956 on Laplace operators on semisimple Lie groups [8] (1). It contains a remarkable result: a description of all irreducible infinite-dimensional representations of complex semisimple Lie groups in Banach spaces. In modern language, Berezin's result can be stated thus: every irreducible representation of a group G is isomorphic to a section of an elementary representation (that is, the representation induced by a one-dimensional character of a Borel

(1) See the list of Berezin's published papers in "Mathematics in the USSR over 40 years (1917-1957)" and "Mathematics in the USSR, 1958-1967"; this list is continued at the end of the obituary.

subgroup). The profundity of this result is clear from the fact that, great efforts notwithstanding, the next step in this direction was not made until some twenty years later. Namely, Zhelobenko and Duflo obtained an explicit classification of all irreducible representations, indicating which sections of elementary representations are equivalent to each other.

In 1956, on the advice of Gel'fand under whose powerful influence he found himself at that time, Berezin went deeply into a study of quantum field theory, and from this time onwards his preoccupation with mathematical physics begins.

In the first period of this work, from the second half of the 50's to the middle of the 60's, Berezin pondered much over questions of spectral theory, in particular, scattering theory for a many-particle Schrödinger operator. Although he obtained few definitive results—a few papers in which one special situation or another is investigated (see [3], [20], [22], [23], [24])—his observations, arguments, and ideas arising as a result of his activity, have had a strong influence on a number of mathematicians and physicists associated with him, and in all they have contributed to the understanding we now possess of the picture of the spectrum and dispersion in the quantum problem of n bodies.

At the beginning of the 60's Berezin completed his work on the formalism of secondary quantization, which he expounded later in his well-known book "The method of second quantization" [25]. This formalism, long used by physicists, is based on the representation of linear operators acting on a so-called Fock space as functions (frequently polynomials) of certain special generators of the algebra of all such operators: the operators of "creation" and "annihilation". Berezin gave an elegant form to his computations, comparing every such polynomial with some polynomial functional on the algebra of functions—in the case of a symmetric Fock space, or with an element of some Grassmann algebra in the case of an antisymmetric Fock space, so that under the actions of the operators (multiplication, conjugation, transformations arising from a canonical change of the original generators, etc.) the corresponding functionals are transformed in one or other of the customary mathematical ways: differentiation, substitution, change of variable, line integration. This method was applied by Berezin himself and by his pupils to a study of certain one-dimensional models of quantum field theory: the Thirring model (for the case of zero mass as well as that of positive mass), the non-linear second-quantized Schrödinger equation and others (see [15], [26], [34]). It should be noted that these papers had a considerable influence on the appearance and development of present-day constructive field theory.

Berezin's work on the second quantization has had a number of important scientific consequences. Interest has reawakened in the old problem of describing representations of the commutation and anticommutation relations (in this connection, see the survey by Golodets in Uspekhi Mat. Nauk **24**:4

(1969), 3–64 = Russian Math. Surveys 24:4 (1969), 1–63). Another theme, born partly out of his work on the second quantization and developed by him over many years, is that of a general understanding of the quantization procedure. Although Berezin was occupied with these questions since the middle of the 60's, the most complete of his conceptions are manifested in a cycle of papers in the years 1973–1975 (see [50], [55], [57], [58]). According to the basic idea of these papers, quantization has the following exact mathematical meaning: the algebra of quantum observations is a deformation of the algebra of classical observations, where Planck's constant serves as the deformation parameter, and the direction of the deformation (the first derivative at zero with respect to the parameter) is given by the Poisson bracket. In the case of a flat phase space this point of view is equivalent to the usual one. In the remaining cases it leads to an interesting new theory. In particular, in his papers in the Izvestiya Akad. Nauk SSSR he considers the case when the phase space is a homogeneous symmetric domain in complex space. He discovered an interesting new effect: the set of possible values of Planck's constant is discrete and bounded above.

Earlier, in the second half of the 60's, Berezin published the article [31] in connection with his work on quantization, a paper in which he studies the representations of operators in Hilbert space by means of various systems of generators in the algebra of such operators (as such generators he considered pq-symbols, qp-symbols, the Weil symbol, and the Wick symbol as used as a rule in the second quantization). We mention that in many of its aspects this paper is close to the theory of pseudodifferential operators, which was new at the time but now plays an important part in mathematical physics. Thus, many important ideas of this theory emerged independently in Berezin's papers ([31], and then in [45], [47]), although unfortunately the significance of his papers from this point of view was by no means recognized straightaway.

Finally, perhaps one of the most important topics in modern mathematical physics whose source also goes back to Berezin's work on the second quantization is the present-day supermathematics: the theory of supergroups and Lie superalgebras and their representations, analysis on supermanifolds. The formal calculus in a Grassmann algebra, which Berezin developed in connection with the formalism of the second quantization in an antisymmetric phase space, led him to the thought that "there exists a non-trivial analogue of analysis in which the rôle of functions is played by elements of a Grassmann algebra" ([25], [18], [28], [40]), that is, analysis in which anticommuting variables appear on equal terms with commuting variables. He made persistent propaganda for this idea and carefully gathered corroborating examples and constructions. The most important of them is the Berezin integral with respect to an anticommuting variable [25], and what was later called the "Berezinian", the analogue of the Jacobian for a change of anticommuting variables [28]. Later, in a joint paper with Kats [40], formal Lie supergroups were introduced and their connection with Lie superalgebras was indicated, by generalizing the exponential map and the theory of Lie groups. Finally, supermanifolds, the last of the important objects in the new theory, were introduced by Leites on the basis of an idea suggested by Berezin [59]. The construction of a supermanifold was carried out in the spirit of modern algebraic geometry: the study of manifolds by means of the local algebras of smooth functions on them, only with the difference that in the case of supermanifolds one must consider superalgebras.

Berezin's pioneering ideas spread up to the middle of the 70's, and groups of supersymmetries, that is, the Lie supergroups of transformations of "super (space-time)" appeared in articles of physicists. Thanks to the labours of Gol'fand and Likhtman, Volkov and Akulov, Bess and Zumino, Ogievetskii and many others, there is reason to hope that a unified field theory can be formulated adequately in the language of superanalysis. This is connected with the following fundamental proposition on the structure of "space-time": "space-time" is a supermanifold whose points are ordinary space-time, and whose group of transformations is the supergroup obtained by extending the Poincaré group by means of odd generators.

In the last years of his life Berezin began writing a book on supermathematics; unfortunately he never finished it. We hope that the completed part will be published and we expect that it will be received with great interest by mathematicians and physicists.

A cherished dream of Berezin, as of many, was the construction of a consistent quantum field theory. He regarded almost all his activities—work on the n-body problem, quantization, superanalysis, etc.—as a preliminary run-up to this difficult problem. He made certain observations and arguments connected with it, such as, for instance, he remarked that the procedure of renormalization in quantum field theory has some features that resemble the theory of extensions of symmetric operators, and he believed for some long time that renormalization can make good sense within the framework of this theory: the original Hamiltonian of the field is well-defined only as a symmetric operator on a suitable set in a Fock space and the true Hamiltonian of the field is obtained as a self-adjoint extension of it. Although this idea turned out to be false, in general, it led to two very good papers: one (with Faddeev) on the δ-shaped interaction of two quantum particles [13], and the other on Lie models [17]. In the second paper Berezin also used Heisenberg's argument that Lie models should be studied in a space with indefinite metric, and in this way he constructed the Hamiltonian of a Lie model as an extension of a symmetric operator in a space with an indefinite metric.

Frequently in the 60's Berezin turned to statistical physics. In 1965 he wrote a joint paper with Sinai on the existence of a phase transition in ferromagnetic lattice systems with finite interaction [33]. In the following years he tried for quite some time to obtain an explicit formula for the statistical sum in the three-dimensional Ising model, using the technique of the seond quantization, which he loved greatly and evidently regarded as a universal tool. Some results he obtained in this direction were published in [37] and [38].

Those are the principal features of Berezin's scientific journey. However, his achievements and their role in mathematical physics can be understood only against the background of his teaching activities, in the widest sense of the word. Patiently he inculcated in the many physicists associated with him a taste for and a love of mathematical thinking, of the elegance of abstract mathematical speculation, and he taught them to apply it all to concrete problems.

For 24 years Berezin worked in the Faculty of Mechanics and Mathematics at the Moscow State University. For almost the whole of this time he conducted a research seminar on mathematical physics (sometimes on his own, sometimes in conjunction with someone else). This seminar was famous among physicists and mathematicians, many young researchers received their education in it, and its participants wrote many excellent articles. In various years he conducted student seminars on representation theory and functional analysis. He delivered Faculty courses on quantum mechanics, statistical physics, quantum field theory, and path integrals. His courses on quantum mechanics and statistical physics were published in mimeographed editions [48], [49]. Berezin intended to improve these courses in time and to make them more accessible to a wide circle of readers, but he did not manage to do so.

He was a man of exceptional modesty, great innate nobility, and absolutely unpretentious.

We believe that the name of Feliks Aleksandrovich Berezin will never be forgotten among mathematicians and physicists. In the future, when all the seeds he has scattered in the scientific world shoot up, he will be remembered often and fondly.

N.N. Bogolyubov, I.M. Gel'fand,
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