metric and the induced metric depends on the embedding. At the Hamiltonian level there are four first class **constraints** (like the super Hamiltonian and supermomentum ones of ADM *canonical gravity*) consequences of the local **diffeomorphism** invariances of the action: they imply the independence of the description from the choice of the $3+1$ splitting of *Minkowski spacetime.*

Through *gauge fixings* one can restrict the hypersurfaces to spacelike hyperplanes and the **constraints** to ten global ones. Moreover, for all the configurations of the isolated system with conserved timelike four-momentum it is possible to restrict the description to those special hyperplanes orthogonal to the configuration momentum. On them only four *first-class constraints* survive: i) one identifies the invariant mass of the isolated system as the effective Hamiltonian; ii) three says that the total three-momentum of the isolated system vanishes (rest-frame conditions). It can be shown that these hyperplanes, named Wigner hyperplanes for covariance reasons, allow the definition and separation of the relativistic canonical center of mass (noncovariant *Newton–Wigner-like position*) of the isolated system and give the intrinsic rest-frame Wigner covariant description of its relative degrees of freedom. The rest-frame instant form of relativistic particles requires a well defined sign of the energy for each one of them, since the intersection of a timelike *worldline* with a spacelike hypersurface (*equal-time surface*) is determined only by three coordinates. Therefore, there is no *mass-shell constraint* $p^2 - m^2 \approx 0$, but two different descriptions for the two disjoint branches of the mass spectrum,

$$p^o \approx \pm\sqrt{m^2 + \vec{p}^2}.$$

For the **spinning particles** of **pseudoclassical mechanics** [5] the semiclassical description of spin is done by using five *Grassmann variables* ξ_μ, ξ_5 generating the *Dirac matrices* $\gamma_5\gamma_\mu$, γ_5 respectively after quantization and one has the two *first-class constraints* $p^2 - m^2 \approx 0$ and

$$p_\mu\xi^\mu - m\xi_5 \approx 0.$$

To get their rest-frame instant form description [6] one has to separate the positive and negative energies. This can be done by boosting at rest the particle (*Chakrabarti representation*) and by describing it with the **constraint**

$$\epsilon - (\pm m) \approx 0,$$

where ϵ is the invariant mass) and its spin only with three *Grassmann variables* (ξ^μ with $p_\mu\xi^\mu = 0$ when we put the gauge fixing $\xi_5 = 0$, giving the *Pauli matrices* after quantization. In this way one gets a description of the $(\frac{1}{2}, 0)$ (for positive energy) and $(0, \frac{1}{2})$ (for negative energy) massive representations of SL(2,C) (relativistic *Pauli particle* without **spinor** equation except in the massless case).

The particle wave equation is the square root *Klein–Gordon equation* for Pauli **spinors** χ and is

$$i\partial_\tau = \sqrt{m^2 + \triangle}\chi.$$

In the case of scalar particles this *pseudodifferential operator* has been studied by Lämmerzahl [7].

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Luca Lusanna

SPINOR-OSCILLATOR REPRESENTATION — A description of spinor and *oscillator representations* from a unifying point of view of **Lie superalgebras**, cf. [2]. These representations are examples of the *Howe duality* [1] which explains many facts of representation theory (classical invariant theory, Capelly identities, parallelism of primitive **differential forms** and *spherical harmonics*, etc.; for more example see [2].)

To describe the representations, consider the Poisson **Lie superalgebra** $\mathfrak{po}(2n\,|\,m)$ whose elements can be labelled by functions $\mathbb{K}[q,p,\Theta]$ over the field $\mathbb{K}$ in $2n$ indeterminates q, p and m indeterminates Θ the **Poisson bracket** $\{\cdot,\cdot\}_{P.b.}$ given by the formula

$$\{f,g\}_{P.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) -$$

$$(-1)^{p(f)}\sum_{j\leq m}\frac{\partial f}{\partial\theta_j}\frac{\partial g}{\partial\theta_j}\ \text{for any } f,g\in\mathbb{C}[p,q,\Theta].$$

For simplicity we consider here $m = 2k$ and subdivide the Θ's into pairs of ξ's and η's; the case of $m = 2k-1$ is even more interesting [2] and reflects a specifics of "super" but formulas and explanations are too long. Introduce: $Q = (q,\xi)$, $P = (p,\eta)$ and set $\deg Q_i = 0$, $\deg P_i = 1$ for all i.

Consider the following quantization, so-called *QP-quantization*, given on linear terms by the formulas:

$$\mathcal{Q}: Q \mapsto \hat{Q}, \quad P \mapsto \hbar\frac{\partial}{\partial Q}, \qquad (*)$$

where $\hat{Q}$ is the operator of left multiplication by Q; an arbitrary monomial should be first rearranged so that the Q's stand first (normal form) and then apply $(*)$ term-wise.

The deformed **Lie superalgebra** $\mathcal{Q}(\mathfrak{po}(2n\,|\,2k))$ is the **Lie superalgebra** of differential operators with polynomial coefficients on $\mathbb{K}^{n|k}$. Actually, it is an analog of $\mathfrak{gl}(V)$. This is most clearly seen for $n = 0$. Indeed,

$$\mathcal{Q}(\mathfrak{po}(0\,|\,2k)) = \mathfrak{gl}(\Lambda^{\boldsymbol{\cdot}}(\xi)) = \mathfrak{gl}(2^{k-1}\,|\,2^{k-1}).$$

In general, for $n \neq 0$, we have

$$\mathcal{Q}(\mathfrak{po}(2n\,|\,2k)) = \text{``}\mathfrak{gl}\text{''}(\mathcal{F}(Q)) = \mathfrak{diff}(\mathbb{K}^{n|k}).$$

The **Lie superalgebra** $\mathfrak{q}(V) = \mathfrak{q}(V, J)$ that preserves the "*complex structure*" in V given by an *odd operator* J such that $J^2 = \pm 1$ (the two algebras corresponding to different signs are isomorphic over $\mathbb{C}$ but not over $\mathbb{R}$), is another, "queer" version of the general linear **Lie algebra**. For $m = 2k - 1$ we consider $\mathfrak{po}(0\,|\,2k-1)$ as a subalgebra of $\mathfrak{po}(0\,|\,2k)$; the quantization sends $\mathfrak{po}(0\,|\,2k-1)$ into $\mathfrak{q}(2^{k-1})$. For $n \neq 0$ the image of $\mathcal{Q}$ is an infinite dimensional version of $\mathfrak{q}$, indeed (for F equal to either $J = i(\theta + \frac{\partial}{\partial\theta})$ with $i^2 = -1$ or for $\Pi = (\theta + \frac{\partial}{\partial\theta})$):

$$\begin{aligned}\mathcal{Q}_F(\mathfrak{po}(2n\,|\,2k-1)) &= \mathfrak{qdiff}(\mathbb{K}^{n|k}) \\ &= \{D \in \mathfrak{diff}(\mathbb{K}^{n|k}) : [d, F] = 0\}.\end{aligned}$$

Setting $\deg_{Lie} f = \deg f - 2$ for any monomial $f \in \mathbb{K}[p, q, \Theta]$, where $\deg p_i = \deg q_i = \deg \Theta_j = 1$ for all i, j, we obtain the standard $\mathbb{Z}$-grading of $\mathfrak{g} = \mathfrak{po}(2n\,|\,m)$; clearly, $\mathfrak{g}_0 = \mathfrak{osp}(m\,|\,2n)$. Let $\mathfrak{g} \longrightarrow \mathfrak{osp}(m\,|\,2n)$ be a representation. The **Lie superalgebras** $\mathfrak{diff}(\mathbb{K}^{n|k})$ and $\mathfrak{qdiff}(\mathbb{K}^{n|k})$ have indescribably many **irreducible representations** even for $n = 0$. But one of the representations, the identity one, in the superspace of functions on $\mathbb{K}^{n|k}$, is the "smallest" one. Moreover, if we consider the **superspace** of $\mathfrak{diff}(\mathbb{K}^{n|k})$ or $\mathfrak{qdiff}(\mathbb{K}^{n|k})$ as the associative superalgebra (denoted $\mathrm{Diff}(\mathbb{K}^{n|k})$ or $\mathrm{QDiff}(\mathbb{K}^{n|k})$), this associative superalgebra has only one **irreducible representation** — the identity one. This representation is called the Fock space.

As is known, the **Lie superalgebras** $\mathfrak{osp}(m\,|\,2n)$ are *rigid* for $(m\,|\,2n) \neq (4\,|\,2)$. Therefore, the through map

$$\mathfrak{h} \longrightarrow \mathfrak{g}_0 = \mathfrak{osp}(m\,|\,2n) \subset \mathfrak{g} = \mathfrak{po}(2n\,|\,m)\mathcal{Q} \longrightarrow \mathfrak{diff}(\mathbb{R}^{n|k})$$

sends any subsuperalgebra $\mathfrak{h}$ of $\mathfrak{osp}(m\,|\,2n)$ (for $(m\,|\,2n) \neq (4\,|\,2)$) into its isomorphic image. One can also embed any *rigid* (e.g., simple) $\mathfrak{h}$ into $\mathfrak{diff}(\mathbb{K}^{n|k})$ directly, not necessarily into $\mathfrak{g}_0$. The irreducible subspace of the Fock space which contains the constants is called the **spinor-oscillator representation** of $\mathfrak{h}$. In particular cases, for $m = 0$ or $n = 0$ this subspace turns into the usual spinor or *oscillator representation*, respectively. We have just given a unified description of them.

Spinor representation is just one of n *fundamental representations* of the finite dimensional orthogonal **Lie algebra** of rank n, the role of **spinor-oscillator representations** grows when we pass to infinite dimensional algebras: all irreducible *highest weight* representations representations of distinguished stringy (super)algebras and the most interesting *fundamental representation* of **Kac–Moody algebras** are constructed in terms of **spinor-oscillator representation** [4,6–8]. **Spinor-oscillator representation** is a key ingredient in calculation of *semi-infinite cohomology* [3,8,9], and manifests itself in realization of **Lie algebras** and superalgebras via creation and annihilation operators [14]. See also [10–13].

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SPINORS — Let us consider D-dimensional *Minkowski spacetime* M_D with flat metric $\eta_{\mu\nu} = \mathrm{diag}(- + \cdots +)$, $\mu, \nu = 0, 1, \ldots, D-1$. The Lorentz group is $SO(1, D-1)$ and the generators of the Lorentz algebra $J_{\mu\nu}$ obey the standard structure relations

$$\begin{aligned}[J_{\mu\nu}, J_{\rho\sigma}] = &-i\eta_{\mu\rho}J_{\nu\sigma} + i\eta_{\nu\rho}J_{\mu\sigma} \\ &- i\eta_{\nu\sigma}J_{\mu\rho} + i\eta_{\mu\sigma}J_{\nu\rho}\end{aligned}$$

The Dirac spinor representation, denoted S_D, is defined in terms of the standard Clifford–Dirac matrices Γ_μ,

$$J_{\mu\nu} = \frac{i}{4}[\Gamma_\mu, \Gamma_\nu], \quad \{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$$

Its (complex) dimension is given by $\dim_C S_D = 2^{[D/2]}$. For D even, the Dirac spinor representation is always