Edge number report 1: state of the art estimates for $n \leqslant 43$.

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Abstract

This first extracted report contains all lower and upper bounds for e-numbers $\mathrm{e}(3, k; n)$, for $n \leqslant 43$, that I know. All but 24 of them are known (exactly). Very little of the proofs is given. A few consequences for upper classical Ramsey number bounds are mentioned.

1 Introduction.

Throughout the years, I have investigated e-numbers, and updated my tables of these and of properties for graphs with edge numbers close to the respective e-number. The results have been collected in the various updated versions of [1]. However, that work is not easily accessible; not only since I have not made it public, but since it is large, and based on a somewhat complex terminology, both for graph objects and for methods for dealing with them.

At present, I'm integrating the consequences of Goedgebeur's and Radziszowski's investigations in [4] into my tables. This is slow work; I have now more or less finished it up to vertex number 43. This has yielded a few improvements, compared both to [4] and to older versions of [1].

I have received some criticism for not making my results more accessible. In this report, I indeed try to present the more recent ones, as regards e-number bounds; but not the further Ramsey graph properties. I believe that this makes it easier to uaccess *the conclusions*; but it makes it harder to reproduce or improve *the proofs*. I outline a few proof examples; they may at least illustrate the 'Ramsey calculus' methods.

Moreover I also discuss upper bounds for e-numbers. This is an area not equally well covered by the literature, I think, and I'm not sure of how good the upper bounds I give here are, compared to the state-of-the-art.

Finally, the terminology is a bit experimentative. I try to make it more conformant to other recent state-of-the-art articles, and (against my instincts) leave a good bit undefined. I'll be very thankful for comments, both on this, and on the factual content of this report.

2 Definitions.

Throughout this work, all graphs $G = (V, E)$ are finite, simple, and undirected; and they are *triangle-free*; i. e., the clique number $\omega(G) \leqslant 2$.

The *second degree* of a vertex v in a graph G is

$$\deg^2(v) = \deg^2_G(v) := \sum_{w \in N(v)} \deg(w)\,,$$

where $N(v)$ is the set of vertices adjacent to v. (The second degree is denoted $Z(v)$ in e. g. [4].) The induced G subgraph on $V \setminus (N(v) \cup \{v\})$ is denoted G_v.

G is an $(i, j; n, e)$*-graph* and an $(i, j; n)$*-graph*, if $\omega(G) < i$, its independence number $\alpha(G) < j$, $n(G) := |V| = n$, and $e(G) := |E| = e$.

For any positive integers i, j, and n, the *e-number* $\mathrm{e}(i, j; n)$ is the minimal number e, such that there are $(i, j; n, e)$-graphs, or ∞, if no $(i, j; n)$-graphs exist. They are of great interest for finding improved bounds of Ramsey numbers

$$\mathrm{R}(i, j) := \min(n : \mathrm{e}(i, j; n) = \infty),$$

but are also of interest in themselves.

In this report, we only discuss the e-numbers $\mathrm{e}(3, j; n)$. For the estimates, we shall use a few linear or 'piecewise linear' functions on two integer variables, namely,

$$f_1(n, k) = \max(0, n - k, 3n - 5k, 5n - 10k, 6n - 13k)\,;$$

$$f_2(n, k) = 8n - 19.5k\,;$$

$$f_3(n, k) = 9n - 23k\,; \text{ and}$$

$$f_4(n, k) = 6.8n - 15.6k\,.$$

Note, that $f_1(n, k) = 6n - 13k$, if $n \geqslant 3k$.

Occasionally, we mention the "linear graph invariant"

$$t(G) := e(G) - 6n(G) + 13\,\alpha(G)\,.$$

$\mathcal{W}_{13;1,5}$ denotes the cyclic graph with 13 vertices (conventionally named u_1, …, u_{13}), and with two vertices forming an edge if the absolute value of their indices counted modulo 13 is either 1 or 5. (This graph very often is denoted H_{13}.)

For other concepts, background, et cetera, see the bibliography. In particular, we shall discuss some graphs given by means of *extension patterns*, which provide recipies for constructing them step-by-step; but neither the patterns and nor the corresponding graphs are formally described here.

3 Known general values.

For $n \leqslant 3.25k + 1.5$, all e-numbers are known. (This indeed includes all $\mathrm{e}(3, k+1; n)$ with $n \leqslant 43$ and $k \geqslant 13$.) To begin with, we have

Proposition 1. *For all positive integers n and k,*

$$\mathrm{e}(3, k+1; n) \geqslant f_1(n, k)\,.$$

The values are exact if and only if $n < \mathrm{R}(3, k+1)$, and moreover either $n \leqslant 3.25k - 1$, or $n = 3.25n$.

For a proof, see e. g. [10]. Note, that part of the result is the fact that $t(G) \geqslant 0$ for all (triangle-free) G.

Lemma 3.1. *Let k and n be positive integers, such that $3k \leqslant n < \mathrm{R}(3, k+1)$, but* $\mathrm{e}(3, k+1; n) > f_1(n, k)$. *Then* $\mathrm{e}(3, k+1; n) = f_1(n, k) + 1 \iff -1 < n - 3.25k < 0$, $\mathrm{e}(3, k+1; n) = f_1(n, k) + 2 \iff 0 < n - 3.25k \leqslant 0.5$, *and* $\mathrm{e}(3, k+1; n) \geqslant f_1(n, k) + 3 \iff 0.5 < n - 3.25k$,

The proof depends on deriving properties for graphs with $t(G) \leqslant 2$. In [1], indeed, all G with $t(G) \leqslant 1$ are characterised, and sufficient restrictions are found for those with $t(G) = 2$. (Actually, the complete characterising of the graphs with $t(G) = 0$ also is the main object of the stand-alone manuscript [2]. The $t(G) = 2$ result partly employs [4].)

Employing some constructions, we find that the lower bound in the last part of lemma 3.1 is exact in a few cases:

Lemma 3.2. *If $3k \leqslant n < \mathrm{R}(3, k+1)$ and $0.5 < n - 3.25k \leqslant 1.5$, then* $\mathrm{e}(3, k+1; n) = f_1(n, k) + 3$.

If $n > 3.25k + 1.5$, and moreover $k \leqslant 12$, then $\mathrm{e}(3, k+1; n) > f_1(n, k) + 3$; and I find it likely that this should hold also for all higher k. Moreover, I guess that

$$\mathrm{e}(3, k+1; n) \geqslant \max(f_2(n, k), f_3(n, k))\,, \tag{1}$$

too; but I am far from being able to prove this. The best general result I have for $n - 3.25k \gg 0$ is

Lemma 3.3. *For any n and k,*

$$\mathrm{e}(3, k+1; n) \geqslant f_4(n, k)\,.$$

(This is contained in [1, proposition 13.5], which is proved by means of a somewhat complicated induction argument).

4 The other values for $n \leqslant 34$.

For $n \leqslant 34$, all $\mathrm{e}(3, k+1; n)$ are known. Actually, only 15 of them are 'sporadic', i. e., not given by the known Ramsey numbers, or in section 3; and they all have $n \geqslant 22$ and $6 \leqslant k \leqslant 9$. Thus, they are included in the following $\mathrm{e}(3, l; n)$ table (where $l = k+1$):

$n \backslash l$	7	8	9	10
22	60	42	30	21
23	∞	49	35	25
24	∞	56	40	30
25	∞	65	46	35
26	∞	73	52	40
27	∞	85	61	45
28	∞	∞	68	51
29	∞	∞	77	58
30	∞	∞	86	66
31	∞	∞	95	73
32	∞	∞	104	81
33	∞	∞	118	90
34	∞	∞	129	99

Note, that all items under an ∞ in a column also are ∞. In the sequel, in each column, just the top ∞ (if any) is printed.

5 The other values and estimates for $35 \leqslant n \leqslant 43$.

In the table, a single value indicates that this is the exact e-value. Two values separated by a dash (–) are the best known lower and upper bounds of the respective e-value. Again, $l = k+1$.

$n \backslash l$	9	10	11	12	13
35	140	107–108	84–85	68	55
36	∞	117–119	92–94	75	60
37		128–(132)	100–103	82	66
38		139–(143)	109–112	89–90	72
39		151–161	119–121	96–98	78
40		161–∞	128–130	103–107	87
41		172–∞	139–(150)	111–116	94
42		∞	149–(160)	120–125	101–102
43			159–(171)	129–134	108–111

The upper bounds within parentheses are rather preliminary; they are achieved by crude constructions, made more or less on the fly, since I am too ignorant to know where to look for the best actually achieved upper bounds. I expect there to have been constructions or computer enumerations around for a while, giving better upper bounds for all five or most of them.

6 Consequences for Ramsey numbers.

By hand calculations or by means of e. g. the matlab programme FRANK ([6])[1], it is fairly easy to check for consequences for upper bounds on Ramsey numbers for any improvement of lower bounds of e-numbers. As compared to the combined values from [4] and older versions of [1], the sharper bounds presented here yield just two improved upper Ramsey number bounds.

It turned out that the improvement of the lower bound for $\mathrm{e}(3,12;43)$ from 128 to 129 was crucial for deducing that

$$\mathrm{R}(3,19) \leqslant 132,$$

as reported in the latest dynamic survey on small Ramsey numbers ([8]).

The improvement of lower $\mathrm{e}(3,11;39)$ bound from 117 ([4]) to 119 suffices to prove that

$$\mathrm{R}(3,16) \leqslant 97\,.$$

This bound is not (yet) included in the dynamic survey.

7 A few proof hints.

7.1 Lower bounds.

Most of the 'sporadic' lower bounds are found in [4]; and/or are direct consequences of lower bounds for smaller independence numbers. The exceptions are the lower bounds for $\mathrm{e}(3,11;35)$, $\mathrm{e}(3,12;38)$, $\mathrm{e}(3,12;39)$, $\mathrm{e}(3,13;41)$, $\mathrm{e}(3,13;42)$, $\mathrm{e}(3,12;43)$, $\mathrm{e}(3,11;39)$, and $\mathrm{e}(3,11;41)$.

The first six of these bounds, as well as the 'general' bounds, depend partly on theoretical classification of some 'lower' graphs, i. e., graphs with lower independence and vertex numbers; likewise, the two last ones depend on computational classification of some lower graphs. In all cases, there is some use of properties deduced for some lower graphs; and the general proof technique is to assume the existence of a graph G with 'offendingly' low $e(G)$, and then to deduce more and more precise conditions for G, until finally a contradiction is achieved. I'll provide a few examples.

First, assume that G is a (3,11;35)-graph with $e(G) \leqslant 83$; whence actually equality must hold. We then successively may prove:

[1]The version of FRANK that I employ includes a test for raising the lower e-number bound in a few cases, where the only formally possible degree distributions all would have to contain either a triangle of low-degree vertices, or a low-degree vertex with too few low-degree neighbours (and thus a too high second degree). In practice, this only may happen, when the unraised e-number bound would be close to, but slightly less than, the e-value for some regular graph. This tweak yielded e. g. $\mathrm{e}(3,13;51) \geqslant 179$.

(a) $\delta(G) > 2$;
(b) $\delta(G) > 3$;
(c) any vertex of degree 4 has at most one neighbour of degree $\geqslant 5$;
(d) G_v has no $\mathcal{W}_{13;1,5}$ component for any vertex of degree 5; and
(e) if $\deg(v) = 5$, then $\deg^2(v) \leqslant 24$.

Property (a) is immediate from the $\mathrm{e}(3, 10; n)$ values.

(b) follows from (a), and from the fact that any $(3, 10; 31)$-graph H with $e(H) \leqslant 74$ has $\delta(H) \geqslant 2$, strictly if $e(H) = 73$; and that there are at most two vertices of degree 2 in H, which (if indeed there are two of them) moreover must be adjacent.

(c) is immediate from (b), and the fact that $\deg^2(v) \leqslant 17$ for any vertex of degree 4.

(e) is an immediate consequence of (d), and of the fact that any $(3, 10; 29, 58)$-graph does contain a $\mathcal{W}_{13;1,5}$ component. On the other hand, (e) directly yields a contradiction, since it means that we could calculate as if $\mathrm{e}(3, 10; 29)$ were at least 59.

This just leaves the deduction of (d) from (b) and (c), which is somewhat less immediate. Assume for a contradiction that $\deg(v) = 5$, and that G_v has a $\mathcal{W}_{13;1,5}$ component. Let $N(v) = \{w_1, \ldots, w_5\}$, and let U be the set of vertices in $\mathcal{W}_{13;1,5}$, which are not adjacent to any w_i; in other words, $U = \{u \in V(\mathcal{W}_{13;1,5}) : \deg_G(u) = 4\}$.

Now, $|U| \leqslant 8$, since U cannot contain an independent 4-set; if it did, any edge between U and $N(v)$ would be redundant (in the sense that removing it from G would leave a graph which also did not contain an independent 11-set), but G can contain neither a redundant edge, nor a $\mathcal{W}_{13;1,5}$ component. Thus, and by inspection of $\mathcal{W}_{13;1,5}$, if U were non-empty, then there were a $u_j \in U$ with at most two neighbours in U, and therefore at least two neighbours of degrees $\geqslant 5$, contradicting (c).

Thus, instead, $U = \emptyset$; i. e., each vertex in $\mathcal{W}_{13;1,5}$ is adjacent to at least one w_i. This makes it possible to apply a "decharging" argument. 'Charge' each u_j with a unit charge, 1; and then 'discharge' each u_j by distributing its charge in equal proportions to its w_i neighbours. The total charge after discharging must stay 13. However, no w_i can receive a charge larger than 2.5; which means that $N(v)$ in total cannot carry a higher charge than 12.5. This is a contradiction; which indeed proves (d).

For a second example, assume that G is a $(3, 11; 41)$-graph with $e(G) = 138$. There are few theoretic ways for such a graph to be 'realised numerically'; in other words, if we let the degree distribution (degree sequence) of the graph be $(n_0, n_1, \ldots, n_{10})$, then there are just a handful possible such sequences, for which the resulting Graver-Yackel defect $\gamma(G)$ would be non-negative (cf. [5] and [4]). In fact, also employing that a single vertex v of degree 8 would have $\deg^2(v) \leqslant 8 \cdot 7 = 56$, and thus a positive defect, and repressing all leading and trailing zeroes in the distributions, we would have one of

$$(11, 30),\ (12, 28, 1),\ (1, 9, 31),\ (2, 7, 32),\ \text{and } (3, 5, 33)$$

as degree distribution, with the total defect $\gamma(G) = 3$, 1, 2, 1, and 0, respectively.

Put $F := \{v \in V : \deg(v) = 7 \text{ and } \deg^2(v) = 48\}$. In other words, F is the set of non-defect vertices of degree 7. Counting directly yields that $|F| \geqslant 27$, in each one of the cases.

For any $f \in F$, G_f is a $(3,10;33,90)$-graph. Now, Goedgebeur and Radziszowski classified all these graphs, and made a list of all 57099 of them available on the *House of Graphs* ([4]). Running the NAUTY ([7]) command `countg --Jd` on this list reveals that any such graph H contains an induced $K_{2,4}$, and has $\delta(H) \geqslant 4$. Moreover, a theoretical analysis shows that for any vertex v with $5 \leqslant \deg(v) \leqslant 7$, either $\delta(G_v) \geqslant 3$, or $\delta(G_v) = 2$ and $\gamma(v) = 3$, or $\gamma(v) > 3$.

Now, choose such an f; if there is a vertex x of degree 8, actually choose $f \in F \cap N(x)$; choose a $K_{2,4} \subset V_f \subset V$, with $V(K_{2,4}) = \{a_1, a_2; b_1, \ldots, b_4\}$ and $\deg(a_1) \leqslant \deg(a_2) \leqslant 7$, say. We now note, that

$$\delta(G_{a_i}) \leqslant \deg(a)_{3-i} - 4, \text{ for } i = 1, 2\,;$$

and employ this in estimating the defects of the a_i.

If $\deg(a_1) = 5$, then $\gamma(a_2) \geqslant 4 > 3 \geqslant \gamma(G)$, a contradiction. Likewise, if $\deg(a_1) = 6$, then $\gamma(a_2) = 3$, whence then $\gamma(a_1) = 0$; whence anyhow

$$6 \leqslant \deg(a_1) \leqslant 7 = \deg(a_2)\,.$$

If $\deg(a_1) = 7$, then both a_1 and a_2 are defective, and the further defects in G sum up to at most 1, whence in particular then $\Delta(G) = 7$. Moreover, if $\deg(a_1) = 7$, then not both a_1 and a_2 may have defects $\geqslant 2$, whence instead then at least one of them has second valency 47, and thus at least five neighbours of degree 7, of which at least four belong to F. Thus, in this case, we may assume that $f' := b_4 \in F$; while if $\deg(a_1) = 6$, then let f' be arbitrarily chosen in $F \cap \operatorname{lk} a_2$. In either case, there is some $K_{2,4}$ in $V_{f'}$, and this would also carry a defect at least 2, which would yield a total defect at least 4 in G, a contradiction.

7.2 Upper bounds.

For $n \leqslant 4k = 4l - 4n$ (but excepting $(n, l) \in \{(17,6),\ (22,7),\ (27,8)\}$), there are constructions, whose connected components either are described by their extension patterns, or are one or the other of two *exceptional graphs*: The cyclic graph $\mathcal{W}_{13;1,5}$ (the unique (3,5;13,26)-graph), and the *twisted tesseract* (a (3,6;16,32)-graph). (The twisted tesseract also is denoted $(2\mathcal{W}_{8;1,4})_{5i}$ in [1]; i. e., it consists of two disjoint copies of $\mathcal{W}_{8;1,4}$, with the i'th vertex in the first copy connected to the $5i$'th one in the second copy by an edge; where indices are taken modulo 8.)

The extension pattern of a graph G of the kind we consider here includes a triangle free graph T, such that

$$e(T) \leqslant 2n(T),$$

$$\alpha(G) = n(T),$$

$$n(G) = 2n(T) + e(T), \text{ and}$$

$$e(G) = n(T) + 2e(T) + \frac{1}{2} \sum_{x \in V(T)} \deg(x)^2\,.$$

This yields that the graphs with only patterned and/or exceptional graphs as components indeed fulfil (1). In fact, for 'most' k and n with $3.25k \leqslant n \leqslant 4k$, we have such graphs realising equality in (1). However, there are some irregularities, for two reasons. First, each $\mathcal{W}_{13;1,5}$ component contributes 4 to the independence number of the graph; and there may not be an integer number of such components that realises equality in (1). Second, in general, for a connected patterned graph G with $3.25\,\alpha(G) \leqslant n(G) \leqslant 4\,\alpha(G)$, equality only can be achieved by having only vertices of degrees 3 and 4 in the pattern graph T (since other degree distributions yield higher $\sum_{V(T)} \deg(x)^2$); which for $(3, 10; 36)$-graphs would force the pattern graph to be 4-regular, on 9 vertices. By inspection, there is no such triangle-free graph; the closest possible degree distribution is (2,5,2) vertices of degrees (3,4,5), respectively.

The upper bound 161 for $\mathrm{e}(3, 10; 39)$ is reported by Goedgebeur and Radziszowski in [4], where it is noted that both they and Exoo have found huge amounts of $(3, 10; 39, 161)$-graphs G, but no $(3, 10; 39)$-graph with a lower number of edges.

For the five upper bounds within parentheses, let L be the regular (3^8)-type lace with constant offsets (1,3), a $(3, 9; 32, 104)$-graph. (Laces are defined and investigated in [1]; they form a special class of patterned graphs.) Its family $(v_1, \ldots, v_8)$ of apices consists of non-adjacent vertices of degree 6, where moreover $\mathrm{dist}(v_i, v_j) \geqslant 3$, if i and j have the same parity. The upper $\mathrm{e}(3, 10; 37)$ ($\mathrm{e}(3, 10; 38)$) bounds are achieved by a 4-extension (5-extension) of H, employing 3 (all 4) of the odd-indexed v_i, respectively; and the upper $\mathrm{e}(3, 11; 41$—$43)$ bounds by making a further extension of one of these, employing the v_i with even indices.

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