Contemporary Mathematics

Local Non-injectivity for Weighted Radon Transforms

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ABSTRACT. A weighted plane Radon transform R_ρ is considered, where $\rho(x, L)$ is a smooth positive function. It is proved that the set of weight functions ρ, for which the map $f \mapsto R_\rho f$ is not locally injective, is dense in the space of smooth positive weight functions.

1. Introduction

We shall consider a weighted plane Radon transform

$$R_\rho f(L) = \int_L f(x)\rho(x, L) ds, \tag{1.1}$$

where L denotes an arbitrary line in the plane, ds is arc length measure on L, and $\rho(x, L)$ is a given, smooth, positive function defined on the set of pairs (x, L) where $x = (x_1, x_2)$ is a point on L. It is well known that R_ρ is not always injective on the set of functions f with compact support [**Bo1**]. On the other hand, if $\rho(x, L)$ is positive and real analytic, it is known that R_ρ is not only injective on compactly supported functions but also *locally injective* in the following sense. Assume that the function f (continuous, say) is supported in the set $\{(x_1, x_2);\, x_2 \geq \delta x_1^2\}$ for some $\delta > 0$ and that $R_\rho f(L) = 0$ for all lines L in a neighborhood of the line $x_2 = 0$; then $f = 0$ in some neighborhood of the origin [**BQ**]. Hence the set of ρ for which R_ρ is locally injective is dense in the set of smooth, positive weight functions. Here we shall show that the set of ρ for which R_ρ is *not locally injective* is also dense (Theorem 1.3). We shall do this by presenting a simplified version of the construction in [**Bo1**] and extending it to a dense set of ρ. By contrast, it is well known that the set of positive ρ for which R_ρ is globally injective is open in the C^1 topology. Indeed, the inverse of R_ρ, if it exists, must be bounded in certain Sobolev norms, and it is a simple fact that the set of operators with bounded inverse must be open. It follows that the set of ρ for which R_ρ is globally injective is open and dense in the set of positive weight functions.

The interest in the mathematical theory of weighted Radon transforms began with the invention of the Single Photon Emission Computed Tomography (SPECT)

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in the 1970s. As is well known, SPECT requires the inversion of the so-called attenuated Radon transform, which corresponds to the case when

$$\rho(x, L) = \exp\big(- \int_{L(x)} \mu(z) ds_z \big), \tag{1.2}$$

where L is an oriented line, $L(x)$ denotes one of the components of $L \setminus \{x\}$, and $\mu(x)$ is a known function (attenuation coefficient) defined on the plane. A breakthrough was achieved by Alexander Bukhgeim et al. [**ABK**] and independently by Roman Novikov [**No**] with proofs of inversion formulas for the attenuated Radon transform R_ρ, where ρ has the form (1.2). Since then many articles on the subject have appeared, for instance [**Na**], [**BS**], [**Ba1**], [**Fi**], [**KB**], [**Ba2**], [**Fo**].

Gindikin recently obtained an inversion formula for a somewhat related, but different class of smooth ρ, [**G**]. For the class of ρ considered in [**G**], local injectivity was proved in [**Bo2**]. For constant ρ local injectivity was first proved by Strichartz [**S**]

It seems not to be known if local injectivity holds for the class of attenuated Radon transforms.

In our study of local injectivity for the plane weighted Radon transform we shall denote the coordinates in $\mathbf{R}^2$ by (x, y) and we shall parametrize the lines as follows: $L(\xi, \eta)$ will denote the line $y = \xi x + \eta$ in the xy plane, and the weight function will be written $\rho(x, \xi, \eta) = \rho(x, L(\xi, \eta))$. Thus R_ρ will be defined by

$$R_\rho f(\xi, \eta) = \int_{\mathbf{R}} f(x, \xi x + \eta) \rho(x, \xi, \eta) dx.$$

Theorem 1.1. *There exists a smooth, positive function $\rho(x, \xi, \eta)$ defined on $\mathbf{R}^3$ and a smooth function $f(x, y)$, supported in $|x| \leq y$ with $(0, 0) \in \operatorname{supp} f$ such that*

$$R_\rho f(\xi, \eta) = 0 \tag{1.3}$$

for all $|\xi| < 1$, $\eta < 1$.

Using Theorem 1.1 we can rather easily construct a counterexample to global injectivity.

Corollary 1.2. ([**Bo1**, Theorem]) *There exists a smooth function f, supported in the unit disk, not identically zero, and a smooth positive weight function $\rho(x, y, L)$ such that $R_\rho f(L) = 0$ for all lines L in the plane.*

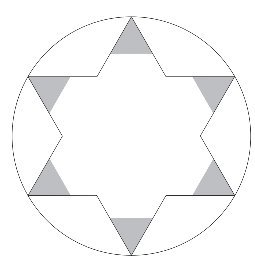

Figure 1

Proof of Corollary 1.2. Note that the weight function cannot be written $\rho(x, L)$ for lines L that are parallel to the y-axis. Let D be the open unit disk and let K be the compact subset of D consisting of the starshaped region in Figure 1 with the six shaded triangular areas omitted. Denote the bottom shaded triangle by T. Using Theorem1.1 we can define a smooth function f in T and a smooth, positive function $\rho(x, y, L)$ such that the bottom point of T is in $\operatorname{supp} f$ and $R_\rho f(L) = 0$ for all lines L intersecting T but not intersecting K. Rotating the coordinate system we can do the same for the other shaded areas. Thus we have defined $f(x, y)$ and $\rho(x, y, L)$ for $(x, y) \notin K$ and lines L not intersecting K such that the support of f meets the boundary of D and $R_\rho f(L) = 0$ for all lines not intersecting

K. The rest of the proof follows a simple idea in [**Bo1**]. Extend f to the entire disk D so that f takes positive as well as negative values on every line intersecting the interior of D. Then for each line L intersecting K it is easy to extend the function $(x, y) \mapsto \rho(x, y, L)$ as a positive function so that the integral of $f(x, y)\rho(x, y, L)$ over L becomes zero. It is also easy to do this in such a way that the function $(x, y, L) \mapsto \rho(x, y, L)$ is smooth. See Lemma 2 in [**Bo1**] for details. This completes the proof. □

Let $\mathcal{W}$ be the set of smooth weight functions $\rho(x, \xi, \eta)$ defined on the set $[-1, 1]^3 \subset \mathbf{R}^3$. On $\mathcal{W}$ we consider the usual C^∞ topology, that is, the topology defined by the family of seminorms p_m, where

$$p_m(\rho) = \sup\{|\partial_\xi^r \partial_\eta^s \partial_x^t \rho|;\, r + s + t \le m\}, \quad m = 0, 1, \dots.$$

Here is the main result of this paper.

THEOREM 1.3. *Let $\rho_0(x, \xi, \eta)$ be an arbitrary smooth, positive function in $\mathcal{W}$. For an arbitrary neighborhood V of ρ_0 there exists a weight function $\rho \in V$ and a function f supported in $|x| \le y$ with $(0, 0) \in \operatorname{supp} f$ such that $R_\rho f(\xi, \eta) = 0$ for $|\xi| < 1$, $\eta < 1$.*

It is obvious that local injectivity for real analytic and positive ρ implies the following *support theorem*, [**BQ**]. Assume f has compact support in $\mathbf{R}^2$, let K be a convex, compact set, and assume $R_\rho f(L) = 0$ for all lines L not intersecting K. Then $f = 0$ in the complement of K.

On the other hand, the argument in the first part of the proof of Corollary 1.2 proves the following.

COROLLARY 1.4. *The set of smooth ρ for which a support theorem does not hold for R_ρ is dense in the set of smooth, positive weight functions.*

2. An example of local non-injectivity

As in [**Bo1**] we shall prove Theorem 1.1 by first constructing the function f and then constructing ρ such that $R_\rho f = 0$. It is clear that f must be highly oscillatory, since f must change sign on every line intersecting the interior of the support of f. To define f we first choose φ and ψ in $C^\infty(\mathbf{R})$ such that

$$\begin{aligned} &\operatorname{supp} \varphi \subset [-a, a], \quad \varphi = 1 \text{ on } [-a_0, a_0] \\ &\operatorname{supp} \psi \subset [1 - b, 1 + b], \quad \psi = 1 \text{ on } [1 - b_0, 1 + b_0], \end{aligned}$$

where $a_0 < a$ and $b_0 < b$ will be chosen later. Set

$$f_k(x, y) = \varphi(2^k x)\psi(2^k y) \cos 4^k x$$

and choose f as the lacunary Fourier series in x

$$f = \sum_{k=0}^{\infty} f_k / k!.$$

It is clear that $f \in C^\infty(\mathbf{R}^2)$.

Lemma 2.1. *One can choose a_0, a, b_0, and b such that $0 < a_0 < a$, $0 < b_0 < b$, and*

(a) $\operatorname{supp} f \subset \{(x,y);\, |x| \le y\}$;

(b) *if the line $L(\xi,\eta)$ meets the support of f_k and $|\xi| \le 1$,*

then $2/5 < 2^k\eta < 8/5$, which can happen for at most two k;

(c) *if $2/3 \le 2^k\eta \le 4/3$, then $\varphi(2^k x)\psi(2^k(\xi x+\eta)) = 1$ for $|x| \le a_0 2^{-k}$.*

Proof. If $\varphi(2^k x)\psi(2^k(\xi x+\eta)) \ne 0$, then $|2^k x| \le a$ and $2^k(\xi x+\eta) \in [1-b, 1+b]$, and if $|\xi| \le 1$ this implies

$$2^k\eta \in [1-b, 1+b] + [-a, a] \subset [1-b-a, 1+b+a].$$

If $4(1-b-a) > 1+b+a)$, i. e. $a+b < 3/5$, this can hold for at most two values of k. This shows that (b) holds if $a+b < 3/5$. On the other hand, if $|\xi| \le 1$, $|2^k x| \le a_0$, and

$$2^k\eta \in [1-b_0+a_0, 1+b_0-a_0],$$

then $2^k(\xi x+\eta) \in [1-b_0, 1+b_0]$, and hence $\varphi(2^k x)\psi(2^k(\xi x+\eta)) = 1$. Thus it is sufficient to choose $b_0 - a_0 \ge 1/3$ to get (c). For instance we can choose $b = 1/2$, $b_0 = 2/5$, $a = 1/16$, and $a_0 = 1/20$. Then $(x,y) \in \operatorname{supp} f_k$ implies $2^k|x| \le a$ and $2^k y \ge 1-b$, hence $y/|x| \ge (1-b)/a = 8$, so (a) certainly holds. $\square$

We shall choose $\rho(x,L) = 1 - c(L)f(x,y)$ for $(x,y) \in L$, that is,

$$\rho(x, L(\xi,\eta)) = \rho(x,\xi,\eta) = 1 - c(\xi,\eta) f(x, \xi x+\eta), \tag{2.1}$$

where $c(\xi,\eta) = c(L(\xi,\eta))$ will be chosen so that

$$\int_L f\rho\, dx = \int_L f dx - c(L)\int_L f^2 dx = 0$$

for all L with $|\xi| < 1$ and $\eta < 1$. This leads to

$$c(L) = \int_L f dx \Big/ \int_L f^2 dx, \quad \text{if } |\xi| < 1 \text{ and } 0 < \eta < 1,$$

and $c(L) = 0$ for all other lines L. The expression for $c(L)$ makes sense, since Lemma 2.1 (c) shows that $\int_L f^2 dx > 0$ when $0 < \eta < 1$. It is clear that $\rho(x,\xi,\eta)$ is smooth for $\eta > 0$. To prove that $\rho(x,\xi,\eta)$ is also smooth near $\eta = 0$ we shall prove that all derivatives of $\rho(x,\xi,\eta)$ tend to zero as η tends to zero.

Lemma 2.2. *For any natural numbers m and p there exists a constant $C_{m,p}$ that is independent of (ξ,η) (and hence independent of k) such that for $2^{-k} \sim \eta$ and $[\xi| < 1$*

$$\big|\partial_\xi^r \partial_\eta^s \int_{L(\xi,\eta)} f dx\big| \le C_{m,p} 2^{-kp}/k!, \quad r+s \le m. \tag{2.2}$$

Proof. Start from the expression

$$\int_{L(\xi,\eta)} f_k dx = \int_{\mathbf{R}} \varphi(2^k x)\psi(2^k(\xi x+\eta)) \cos 4^k x\, dx$$

and make q partial integrations, which gives, if for instance q is even,

$$\int_{L(\xi,\eta)} f_k dx = (-1)^{q/2} 4^{-kq} \int_{\mathbf{R}} \cos 4^k x\ \partial_x^q \big(\varphi(2^k x)\psi(2^k(\xi x+\eta))\big) dx.$$

If q is odd, we get a similar expression with sin instead of cos. The absolute value of the integral can be estimated by $C_q 2^{kq}$, which gives

$$|\int_{L(\xi,\eta)} f_k dx| \leq C_q 2^{-kq}.$$

Applying the derivative $\partial_\xi^r \partial_\eta^s$ adds a factor $(2^k|\xi|)^r 2^{sk} \leq 2^{k(r+s)}$, which gives

$$|\partial_\xi^r \partial_\eta^s \int_{L(\xi,\eta)} f_k dx| \leq C'_{m,q} 2^{k(m-q)} \leq C'_{m,m+p} 2^{-kp}, \quad r+s \leq m, \tag{2.3}$$

if we choose $q = m + p$. To finish the proof of the lemma we use Lemma 2.1 (b), which shows that

$$\int_{L(\xi,\eta)} f dx = \int_{L(\xi,\eta)} \Big(\frac{f_k}{k!} + \frac{f_j}{j!}\Big) dx$$

with $j = k \pm 1$ for $2^{-k} \sim \eta$, if $|\xi| \leq 1$. □

Lemma 2.3. *We have the estimate*

$$\int_{L(\xi,\eta)} f^2 dx \geq a_0 2^{-k-3}/(k!)^2, \tag{2.4}$$

if $2^k\eta \in [2/3, 4/3]$.

Proof. Let $L(\xi,\eta)$ be given with $|\xi| < 1$ and $0 < \eta < 1$. Choosing k as in Lemma 2.1 (c) we have $f_k = \cos 4^k x$ on L for $|x| \leq a_0 2^{-k}$ and by (b) we know that $f = f_k/k! + f_j/j!$ on L, where $j = k \pm 1$. If f_k and f_j have the same sign, then $f^2 \geq f_k^2/(k!)^2$. Since this must occur on $I = [-a_0 2^{-k}, a_0 2^{-k}]$ in a number of quarterperiods of $\cos 4^k x$ with total length at least slightly less than half of the length of I, we can certainly say that

$$\int_{L(\xi,\eta)} f^2 dx \geq \frac{1}{(k!)^2}\frac{1}{4}\int_I \cos^2 4^k x\, dx \geq \frac{1}{(k!)^2}\frac{1}{4}\cdot\frac{1}{4}|I| = \frac{a_0}{(k!)^2}\, 2^{-k-3}.$$

□

Lemma 2.4. *If* $2^k\eta \in [2/3, 4/3]$ *and* $|\xi| < 1$, *then*

$$\big|\partial_\xi^r \partial_\eta^s \int_{L(\xi,\eta)} f^2 dx\big| \leq \frac{C_p}{(k!)^2} 2^{kp}, \quad r+s \leq p, \tag{2.5}$$

where C_p *depends only on* p.

Proof. Differentiating the expression

$$\int_{L(\xi,\eta)} f_k^2 dx = \int_{\mathbf{R}} \varphi(2^k x)^2 \psi(2^k(\xi x + \eta))^2 \cos^2 4^k x\, dx$$

we immediately obtain the estimate

$$\big|\partial_\xi^r \partial_\eta^s \int_{L(\xi,\eta)} f_k^2 dx\big| \leq C_p 2^{kp}, \quad r+s \leq p. \tag{2.6}$$

By Lemma 2.1 (c) we know that

$$\int_{L(\xi,\eta)} f^2 dx = \int_{\mathbf{R}} \big(\frac{f_k}{k!} + \frac{f_j}{j!}\big)^2 dx$$

with $j = k \pm 1$, if k is chosen so that $2^k\eta \in [2/3, 4/3]$. Applying (2.6) and a similar estimate for derivatives of $f_k f_{k\pm1}$ we obtain (2.5). □

Lemma 2.5. *If $2^k\eta \in [2/3, 4/3]$ and $|\xi| < 1$, then*

$$\big|\partial_\xi^r \partial_\eta^s \frac{1}{\int_{L(\xi,\eta)} f^2 dx}\big| \le C_p (k!)^2 2^{kp}, \quad r + s \le p, \tag{2.7}$$

where C_p depends only on p.

For the proof of Lemma 2.5 we shall use the following elementary fact.

Lemma 2.6. *Let h be a smooth function on an open set $U \subset \mathbf{R}^n$ such that $h \ge 1$ on U and*

$$\sup_{x \in U,\, |\alpha| \le p} |\partial^\alpha h(x)| \le M_p, \quad p = 0, 1, \ldots.$$

Let N_p be a logarithmically convex sequence such that $N_p \ge M_p$ and $N_0 \ge 1$. Then there exist constants A_p that depend only on p and on the dimension n such that

$$\sup_{x \in U,\, |\alpha| \le p} \Big|\partial^\alpha \frac{1}{h(x)}\Big| \le A_p N_0^p N_p, \quad p = 0, 1, \ldots.$$

Proof. An arbitrary partial derivative $\partial^\alpha(1/h)$ of order $|\alpha| = p$ is equal to a sum of terms of the form

$$C_\alpha \frac{1}{h^{q+1}} \prod_{\nu=1}^{q} \partial^{\beta_\nu} h,$$

where $\sum \beta_\nu = \alpha$ (α and β_ν are multi-index) and $q \le |\alpha|$. By the assumption we have $|\partial^{\beta_\nu} h| \le M_{|\beta_\nu|} \le N_{|\beta_\nu|}$, and by logarithmic convexity $N_s \le N_0^{(p-s)/p} N_p^{s/p} \le N_0 N_p^{s/p}$, hence

$$\prod_{\nu=1}^{q} |\partial^{\beta_\nu} h| \le N_0^p N_p.$$

The sum of all the coefficients C_α depends only on p and the dimension n. This completes the proof. $\square$

Proof of Lemma 2.5. By Lemma 2.3 the function

$$h(\xi, \eta) = a_0^{-1} 2^{k+3} (k!)^2 \int_{L(\xi,\eta)} f^2 dx \tag{2.8}$$

satisfies $h(\xi, \eta) \ge 1$ if $2^k\eta \in [2/3, 4/3]$. Let B_p be a logarithmically convex sequence satisfying $B_0 \ge 1$ and $B_p \ge 8C_p/a_0$, where C_p are the constants in Lemma 2.4. Applying Lemma 2.6 with $N_p = B_p 2^{k(p+1)}$ we obtain

$$\big|\partial_\xi^s \partial_\eta^t \frac{1}{h(\xi, \eta)}\big| \le A_p N_0^p N_p 2^{(p+1)k}, \quad s + t \le p.$$

Taking into account (2.8) we obtain (2.7). $\square$

End of proof of Theorem 1.1. It remains only to show that the weight function $\rho(x, \xi, \eta)$ defined by (2.1) is smooth on $\{(x, \xi, \eta) \in \mathbf{R}^3;\, |\xi| < 1,\, \eta < 1\}$. We first prove that there are constants $C_{m,p}$ that depend only on m and p such that

$$\big|\partial_\xi^r \partial_\eta^s c(\xi, \eta)\big| \le C_{m,p} 2^{-kp} k!, \quad r + s \le m, \tag{2.9}$$

if k is chosen so that $2^k\eta \in [2/3, 4/3]$. Writing $c(\xi, \eta)$ as a product

$$c(\xi, \eta) = \int_{L(\xi,\eta)} f dx \cdot \frac{1}{\int_{L(\xi,\eta)} f^2 dx}$$

we express an arbitrary derivative of order m of $c(\xi,\eta)$ using Leibnitz' formula. In each term in the resulting sum we estimate a derivative of order $\le m$ of $1/\int_{L(\xi,\eta)} f^2 dx$ by $C_m(k!)^2 2^{km}$ using Lemma 2.5. Replacing p by $p+m$ in Lemma 2.2 we can estimate an arbitrary derivative of order $\le m$ of $\int_{L(\xi,\eta)} f dx$ by

$$C_{m,p} 2^{-k(m+p)}/k!.$$

Since the sum of all the coefficients in Leibnitz' formula depends only on m, the estimate (2.9) follows.

To finish the proof of the theorem we use the fact that

$$\begin{aligned} 1-\rho(x,\xi,\eta) &= c(\xi,\eta) f(x,\xi x+\eta) \\ &= c(\xi,\eta)\Big(\frac{f_k(x,\xi x+\eta)}{k!}+\frac{f_j(x,\xi x+\eta)}{j!}\Big) \end{aligned} \tag{2.10}$$

with $j = k \pm 1$ if k is chosen as indicated. We have to prove that all derivatives of $\rho(x,\xi,\eta)$ tend to zero as η tends to $+0$. An arbitrary derivative of order $\le m$ of $f_k(x,\xi x+\eta)$ can be estimated by $C_m 4^{km}$. By (2.9) it follows that an arbitrary derivative of order $\le m$ of $c(\xi,\eta) f_k(x,\xi x+\eta)/k!$ can be estimated by

$$C_{m,p} 2^{-kp} C_m 4^{km}, \quad \text{for } \eta \sim 2^{-k}. \tag{2.11}$$

If we choose $p = 2m+1$ this expression becomes $\le C\eta$, which proves the claim. This completes the proof of Theorem 1.1. □

3. A dense set of non locally injective ρ

The proof of Theorem 1.3 depends on the simple observations that we can make the coefficient $c(\xi,\eta)$ in (2.1) arbitrarily small by making the functions f_k sufficiently oscillatory, and that nothing is changed in the arguments above if we replace the constant function 1 by an arbitrary smooth function $\rho_0(x,\xi,\eta)$.

Define $f = \sum f_k/k!$ as before, where f_k now depends on a parameter λ, that will have a fixed (large) value ≥ 1:

$$f_k(x,y,\lambda) = \varphi(2^k x)\psi(2^k y)\cos(4^k \lambda x), \tag{3.1}$$

and choose

$$\rho(x,\xi,\eta) = \rho_0(x,\xi,\eta) - c(\xi,\eta,\lambda) f(x,\xi x+\eta,\lambda),$$

where

$$c(\xi,\eta,\lambda) = \int_{L(\xi,\eta)} f\rho_0\, dx \Big/ \int_{L(\xi,\eta)} f^2 dx$$

for $0<\eta<1$ and $c=0$ for $\eta \le 0$. Then it is clear that $\int_L f\rho\, dx = 0$ for all relevant lines L, so we only need to prove that ρ is smooth and that $\rho \in V$ if λ is sufficiently large.

Lemma 3.1. *Let $\rho_0(x,\xi,\eta)$ be an arbitrary smooth function on $\mathbf{R}^3$. For any natural numbers m and p there exists a constant $C_{m,p}$ that is independent of (ξ,η) (and hence independent of k) and is independent of $\lambda \ge 1$ such that for $2^{-k} \sim \eta$ and $[\xi| < 1$*

$$\Big|\partial_\xi^r \partial_\eta^s \int_{L(\xi,\eta)} f\rho_0\, dx\Big| \le C_{m,p} 2^{-kp}\lambda^{-p}/k!, \quad r+s \le m. \tag{3.2}$$

Proof. The proof is completely parallel to the proof of Lemma 2.2, so we only need to point out how the parameter λ comes up in the formulas. It will clearly be enough to prove

$$\big|\partial_\xi^r \partial_\eta^s \int_{L(\xi,\eta)} f_k \rho_0 \, dx\big| \le C_{m,p} 2^{-kp} \lambda^{-p}, \quad r+s \le m, \tag{3.3}$$

instead of (2.3). Making q partial integrations in the expression for $\int_{L(\xi,\eta)} f_k \rho_0 \, dx$ we obtain if q is even

$$\begin{aligned}(-1)^{q/2} &\int_{L(\xi,\eta)} f_k \rho_0 \, dx \\ = (4^k \lambda)^{-q} &\int_{\mathbf{R}} \cos 4^k x \; \partial_x^q \big(\varphi(2^k \lambda x) \psi(2^k(\xi x + \eta)) \rho_0(x,\xi,\eta)\big) dx.\end{aligned}$$

Applying the derivative $\partial_\xi^r \partial_\eta^s$ with $r+s \le m$ to this expression we obtain

$$\big|\partial_\xi^r \partial_\eta^s \int_{L(\xi,\eta)} f_k \rho_0 \, dx\big| \le (4^k \lambda)^{-q} C_{m+q} 2^{k(m+q)} = C_{m+q} 2^{km} 2^{-kq} \lambda^{-q},$$

and choosing $q = m + p$ completes the proof. □

End of proof of Theorem 1.3. We have to show that, for arbitrary given N and $\delta > 0$, we can choose λ so that

$$\sup_{|\alpha| \le N, \, |\xi| < 1, \, \eta < 1} |\partial^\alpha (\rho - \rho_0)| < \delta, \tag{3.4}$$

where we have written $\partial^\alpha = \partial_\xi^r \partial_\eta^s \partial_x^t$, $\alpha = (r,s,t)$. The estimates for the derivatives of $\rho(x,\xi,\eta) - \rho_0(x,\xi,\eta) = c(\xi,\eta,\lambda) f(x, \xi x + \eta, \lambda)$ will be exactly the same as in the proof of Theorem 1.1 apart from the presence of the factor λ^{-p} in the right hand side of (3.2). Indeed, the estimates for $\int_{L(\xi,\eta)} f^2 dx$ and its inverse will be unaffected by the parameter λ, so for $c(\xi,\eta,\lambda)$ we get by means of Lemma 3.1 and Lemma 2.5

$$\big|\partial_\xi^r \partial_\eta^s c(\xi,\eta,\lambda)\big| \le \frac{C_{m,p} 2^{-kp} \lambda^{-p}}{k!} \, 2^{km} (k!)^2 \le C'_{m,p} 2^{k(m-p)} \lambda^{-p} k!, \quad r+s \le m$$

instead of (2.9). For the derivatives of $f_k(x, \xi x + \eta, \lambda)$ we have

$$|\partial_\xi^r \partial_\eta^s \partial_x^t f_k(x, \xi x + \eta, \lambda)| \le C_m 2^{(r+s)k} (4^k \lambda)^t \le C_m 2^{2mk} \lambda^m, \quad r+s+t \le m.$$

Thus, taking into account (2.10) we obtain with a new constant $C_{m,p}$

$$|\partial^\alpha (\rho(x,\xi,\eta) - \rho_0(x,\xi,\eta))| \le C_{m,p} 2^{3mk} 2^{-pk} \lambda^{-p}, \quad |\alpha| \le m.$$

With $m = N$ and $p = 3N$ this gives $|\partial^\alpha(\rho - \rho_0)| \le C_{N,3N} \lambda^{-3N}$, which we can make as small as we please by taking λ sufficiently large. This completes the proof of Theorem 1.3. □

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