Rational homotopy theory

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Abstract

These are lecture notes for a course on rational homotopy theory given at the University of Copenhagen in the fall of 2012.

Contents

1 Introduction

By the fundamental work of Serre, the homotopy groups $\pi_n(X)$ of a simply connected finite complex X are finitely generated abelian groups. These may therefore be decomposed as

$$\pi_n(X) = \mathbb{Z}^r \oplus T,$$

where r is the rank and T is the torsion subgroup, which is finite. The torsion is mysterious; there is no non-contractible simply connected finite complex X for which the torsion in $\pi_n(X)$ is known for all n. On the contrary, the rank can often be determined. For example, Serre showed that $\pi_k(S^n)$ is finite except when $k = n$ or when n is even and $k = 2n-1$. In the exceptional cases the rank is 1. This observation is fundamental for rational homotopy theory. The idea is that by 'ignoring torsion' one should obtain a simpler theory.

On the level of homotopy groups, one way of ignoring torsion is to tensor with the rational numbers:

$$\pi_n(X) \otimes \mathbb{Q} = \mathbb{Q}^r.$$

This can also be done on the space level. There is a construction $X \mapsto X_{\mathbb{Q}}$, called *rationalization*, with the property that $\pi_n(X_{\mathbb{Q}}) \cong \pi_n(X) \otimes \mathbb{Q}$. Two spaces X

and Y are said to be rationally homotopy equivalent, written $X \sim_{\mathbb{Q}} Y$, if their rationalizations $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ are homotopy equivalent. Rational homotopy theory is the study of spaces up to rational homotopy equivalence.

There are two seminal papers in the subject, Quillen's [20] and Sullivan's [25]. Both associate to a simply connected topological space X an algebraic object, a *minimal model* M_X. Sullivan uses commutative differential graded algebras, whereas Quillen uses differential graded Lie algebras. The key property of the minimal model is that

$$X \sim_{\mathbb{Q}} Y \text{ if and only if } M_X \text{ and } M_Y \text{ are } \textit{isomorphic.}$$

Thus, the minimal model solves the problem of classifying simply connected spaces up to rational homotopy equivalence. Moreover, the minimal model can often be determined explicitly, allowing for calculations of rational homotopy invariants.

These lecture notes are based on material from various sources, most notably [1, 5, 9, 10, 11, 12, 20, 25, 26, 27]. Needless to say, any mistakes in the text are my own. We hope that these notes may serve as a more concise alternative to [5]. For a nice brief introduction to rational homotopy theory and its interactions with commutative algebra, see [12]. An account of the history of rational homotopy theory can be found in [13].

2 Eilenberg-Mac Lane spaces and Postnikov towers

In this section, we will recall the basic properties of Eilenberg-Mac Lane spaces and Postnikov towers of simply connected spaces. We refer the reader to [10] for more details and proofs.

2.1 Eilenberg-Mac Lane spaces

Let A be an abelian group and $n \geq 1$. The Eilenberg-Mac Lane space $K(A, n)$ is a connected CW-complex that is determined up to homotopy equivalence by the requirement that there exists a natural isomorphism

$$[X, K(A, n)] \cong \widetilde{\mathrm{H}}^n(X; A)$$

for all CW-complexes X. In particular, if we plug in $X = S^n$, we see that

$$\pi_n(K(A, n)) = A, \quad \pi_k(K(A, n)) = 0, \quad k \neq n,$$

and in fact already this property characterizes $K(A, n)$ up to homotopy equivalence.

Now, let X be a connected space. The spaces $\prod_n K(\pi_n(X), n)$ and X have the same homotopy groups, but it is not true in general that they have the same homotopy type. However, it is true that X has the homotopy type of a 'twisted' product of the Eilenberg-Mac Lane spaces $K(\pi_n(X), n)$. Let us make this more precise.

2.2 Postnikov towers

A *Postnikov tower* for a simply connected space X is a tower of fibrations

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_3 \longrightarrow X_2$$

together with compatible maps $X \to X_n$ such that

- $\pi_k(X) \to \pi_k(X_n)$ is an isomorphism for $k \leq n$,
- $\pi_k(X_n) = 0$ for $k > n$.

It follows that the fiber of $X_n \to X_{n-1}$ is an Eilenberg-Mac Lane space of type $K(\pi_n(X), n)$. In other words, there is a homotopy fiber sequence

$$K(\pi_n(X), n) \to X_n \to X_{n-1}.$$

In this sense, X_n is a twisted product of X_{n-1} and $K(\pi_n(X), n)$. The space X_n is sometimes referred to as the 'n^{th} Postnikov section of X'.

Simply connected CW-complexes admit Postnikov towers with the further property that each fibration $X_n \to X_{n-1}$ is principal, i.e., a pullback of the path fibration over an Eilenberg-Mac Lane space,

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & PK(\pi_n(X), n+1) \arrow[d] \\ X_{n-1} \arrow[r, "k^{n+1}"] & K(\pi_n(X), n+1), \end{tikzcd} \tag{1}$$

along a certain map k^{n+1}. The map k^{n+1} represents a class

$$[k^{n+1}] \in [X_{n-1}, K(\pi_n(X), n+1)] = \mathrm{H}^{n+1}(X_{n-1}; \pi_n(X)),$$

called the $(n+1)$-st *k-invariant*. The collection of homotopy groups $\pi_n(X)$ together with all k-invariants k^{n+1} are enough to reconstruct X up to homotopy equivalence. The space X is homotopy equivalent to the product of Eilenberg-Mac Lane spaces $\prod_n K(\pi_n(X), n)$ if and only if all k-invariants are trivial.

3 Homotopy theory modulo a Serre class of abelian groups

Rational homotopy theory is homotopy theory 'modulo torsion groups'. In this section we will make precise what it means to do homotopy theory 'modulo $\mathscr{C}$', where $\mathscr{C}$ is some class of abelian groups. This idea is due to Serre [22].

3.1 Serre classes

Consider the following conditions on a class of abelian groups $\mathscr{C}$.

(i) Given a short exact sequence of abelian groups

$$0 \to A' \to A \to A'' \to 0,$$

the group A belongs to $\mathscr{C}$ if and only if both A' and A'' belong to $\mathscr{C}$.

(i)' Given an exact sequence of abelian groups

$$A \to B \to C \to D \to E,$$

if A, B, D and E belong to $\mathscr{C}$, then C belongs to $\mathscr{C}$.

Definition 3.1. A non-empty class of abelian groups $\mathscr{C}$ is satisfying (i) is called a *Serre class.* A non-empty class of abelian groups $\mathscr{C}$ satisfying condition (i)' is called a *Serre' class*[1].

Obvious examples of Serre classes are the class of all abelian groups, and the class of trivial groups.

Exercise 3.2.
1. Verify that the following are Serre classes.
   (a) Finitely generated abelian groups.
   (b) Torsion abelian groups.
   (c) Finite abelian groups.
2. Show that every Serre class is a Serre' class.
3. An abelian group A is called *uniquely divisible* if the multiplication map $A \xrightarrow{\cdot n} A$ is an isomorphism for every non-zero integer n.
   (a) Show that an abelian group A is uniquely divisible if and only if the canonical map $A \to A \otimes \mathbb{Q}$ is an isomorphism.
   (b) Show that the class of uniquely divisible groups is a Serre' class, but not a Serre class.
4. Let $\mathscr{C}$ be a Serre' class. Verify the following statements.
   (a) Given a short exact sequence of abelian groups

   $$0 \to A' \to A \to A'' \to 0,$$

   if two out of A', A, A'' belong to $\mathscr{C}$, then so does the third.
   (b) If C_* is a chain complex of abelian groups with $C_n \in \mathscr{C}$ for all n, then $\mathrm{H}_n(C_*) \in \mathscr{C}$ for all n.
   (c) If $0 = A_{-1} \subset A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset A_n = A$ is a filtration of an abelian group A such that the filtration quotients A_p/A_{p-1} belong to $\mathscr{C}$ for all p, then A belongs to $\mathscr{C}$.

We will need the following technical conditions, for reasons that will become clear soon.

(ii) If A and B belong to $\mathscr{C}$, then so do $A \otimes B$ and $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$.

(iii) If A belongs to $\mathscr{C}$, then so does $\mathrm{H}_k(K(A, n))$ for all $k, n > 0$.

[1] This is not standard terminology.

3.2 Hurewicz and Whitehead theorems modulo $\mathscr{C}$

Theorem 3.3. *Let $\mathscr{C}$ be a Serre' class of abelian groups satisfying (ii) and (iii), and let X be a simply connected space. If $\pi_k(X) \in \mathscr{C}$ for all k, then $\mathrm{H}_k(X) \in \mathscr{C}$ for all $k > 0$.*

Proof. Let us write $\pi_n := \pi_n(X)$ to ease notation. Let $\{X_n\}$ be a Postnikov tower of X. We will prove by induction on n that $\mathrm{H}_k(X_n) \in \mathscr{C}$ for all $k > 0$ and all $n \geq 2$. Then we are done because $\mathrm{H}_k(X) \cong \mathrm{H}_k(X_k)$. To start the induction, the fact that $X_2 = K(\pi_2, 2)$ and $\pi_2 \in \mathscr{C}$ implies that $\mathrm{H}_k(X_2) \in \mathscr{C}$ for all $k > 0$ by (iii). For the inductive step, suppose that $\mathrm{H}_k(X_{n-1}) \in \mathscr{C}$ for all $k > 0$. Then consider the Serre spectral sequence of the fibration

$$K(\pi_n, n) \to X_n \to X_{n-1}.$$

It has E^2-term

$$\begin{aligned} E^2_{p,q} &= \mathrm{H}_p(X_{n-1}; \mathrm{H}_q(K(\pi_n, n))) \\ &\cong \mathrm{H}_p(X_{n-1}) \otimes \mathrm{H}_q(K(\pi_n, n)) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}_{p-1}(X_{n-1}), \mathrm{H}_q(K(\pi_n, n)). \end{aligned}$$

The isomorphism comes from the universal coefficient theorem. Since $\pi_n \in \mathscr{C}$ and $\mathrm{H}_k(X_{n-1}) \in \mathscr{C}$ for all $k > 0$, it follows from (i)', (ii) and (iii) that $E^2_{p,q} \in \mathscr{C}$ for all $(p, q) \neq (0, 0)$. Since $E^{r+1} = \mathrm{H}_*(E^r, d_r)$, and $E^\infty_{p,q} = E^r_{p,q}$ for $r \gg p, q$, it follows from Exercise 3.2 that $E^\infty_{p,q} \in \mathscr{C}$ for all $(p, q) \neq (0, 0)$. The groups $E^\infty_{p,q}$ are the quotients F_p/F_{p-1} in a filtration

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{p+q} = \mathrm{H}_{p+q}(X_n).$$

It follows from Exercise 3.2 that $\mathrm{H}_k(X_n) \in \mathscr{C}$ for all $k > 0$. □

Theorem 3.4. *Let $\mathscr{C}$ be a Serre' class of abelian groups satisfying (ii) and (iii), and let X be a simply connected space. If $\mathrm{H}_k(X) \in \mathscr{C}$ for all $k > 0$, then $\pi_k(X) \in \mathscr{C}$ for all $k > 0$.*

Proof. Again, let $\pi_n := \pi_n(X)$ and let $\{X_n\}$ be a Postnikov tower of X. We will prove by induction that $\pi_k(X_n) \in \mathscr{C}$ for all k and all $n \geq 2$. Then we are done because $\pi_k(X) = \pi_k(X_k)$. Since $X_2 = K(\pi_2, 2)$ and $\pi_2 \cong \mathrm{H}_2(X) \in \mathscr{C}$ by the ordinary Hurewicz theorem, we have $\pi_k(X_2) \in \mathscr{C}$ for all k. Suppose by induction that $\pi_k(X_{n-1}) \in \mathscr{C}$ for all k. Then $\mathrm{H}_k(X_{n-1}) \in \mathscr{C}$ for all $k > 0$ by Theorem 3.3. We can convert the map $X \to X_{n-1}$ into an inclusion by using the mapping cylinder. Since $\pi_k(X_{n-1}) = 0$ for $k \geq n$, it follows from the long exact homotopy sequence of the pair (X_{n-1}, X),

$$\cdots \to \pi_k(X) \to \pi_k(X_{n-1}) \to \pi_k(X_{n-1}, X) \to \pi_{k-1}(X) \to \cdots,$$

that $\pi_{n+1}(X_{n-1}, X) \cong \pi_n(X)$. The map $X \to X_{n-1}$ is n-connected, so by the relative Hurewicz theorem $\pi_{n+1}(X_{n-1}, X) \cong \mathrm{H}_{n+1}(X_{n-1}, X)$. Finally, consider the long exact sequence in homology

$$\mathrm{H}_{n+1}(X) \to \mathrm{H}_{n+1}(X_{n-1}) \to \mathrm{H}_{n+1}(X_{n-1}, X) \to \mathrm{H}_n(X) \to \mathrm{H}_n(X_{n-1}).$$

Since $\mathrm{H}_k(X)$ and $\mathrm{H}_k(X_{n-1})$ belong to $\mathscr{C}$ for all $k > 0$, it follows from the exercise that the middle term belongs to $\mathscr{C}$. Hence, $\pi_n(X) \cong \pi_{n+1}(X_{n-1}, X) \cong \mathrm{H}_{n+1}(X_{n-1}, X) \in \mathscr{C}$. □

A homomorphism $f\colon A \to B$ is called a *$\mathscr{C}$-monomorphism* if $\ker f$ belongs to $\mathscr{C}$, a *$\mathscr{C}$-epimorphism* if $\operatorname{coker} f$ belongs to $\mathscr{C}$, and a *$\mathscr{C}$-isomorphism* if it is both a $\mathscr{C}$-monomorphism and a $\mathscr{C}$-epimorphism.

Theorem 3.5 (Hurewicz Theorem mod $\mathscr{C}$)**.** *Let $\mathscr{C}$ be a Serre' class of abelian groups satisfying (ii) and (iii), and let X be a simply connected space. The following are equivalent:*

1. *$\pi_k(X)$ belongs to $\mathscr{C}$ for all $k < n$.*
2. *$\widetilde{\mathrm{H}}_k(X)$ belongs to $\mathscr{C}$ for all $k < n$.*

In this situation, there is an exact sequence

$$K \to \pi_n(X) \to \mathrm{H}_n(X) \to C \to 0,$$

where K and C belong to $\mathscr{C}$. In particular, if $\mathscr{C}$ is a Serre class, then the Hurewicz homomorphism $\pi_n(X) \to \mathrm{H}_n(X)$ is a $\mathscr{C}$-isomorphism.

Proof. Consider the Postnikov section X_{n-1}. By definition, the map $X \to X_{n-1}$ induces an isomorphism on π_k, and hence on $\widetilde{\mathrm{H}}_k$, for $k < n$. Hence, if $\pi_k(X) \in \mathscr{C}$ for $k < n$, then $\pi_k(X_{n-1}) \in \mathscr{C}$ for all k. But then $\widetilde{\mathrm{H}}_k(X_{n-1}) \in \mathscr{C}$ for all k by Theorem 3.3. Since $\mathrm{H}_k(X) \cong \mathrm{H}_k(X_{n-1})$ for $k < n$, it follows that $\widetilde{\mathrm{H}}_k(X) \in \mathscr{C}$ for all $k < n$. The claim about the Hurewicz homomorphism follows by considering the following commutative diagram with exact rows

$$\begin{tikzcd}
= 0 & & & = 0 & = 0 \\
\pi_{n+1}(X_{n-1}) \arrow[r] \arrow[d] & \pi_{n+1}(X_{n-1}, X) \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi_n(X) \arrow[r] \arrow[d] & \pi_n(X_{n-1}) \arrow[r] \arrow[d] & \pi_n(X_{n-1}, X) \arrow[d] \\
\mathrm{H}_{n+1}(X_{n-1}) \arrow[r] & \mathrm{H}_{n+1}(X_{n-1}, X) \arrow[r] & \mathrm{H}_n(X) \arrow[r] & \mathrm{H}_n(X_{n-1}) \arrow[r] & \mathrm{H}_n(X_{n-1}, X) \\
\in \mathscr{C} & & & \in \mathscr{C} & = 0
\end{tikzcd}$$

The vertical maps are the Hurewicz homomorphisms. $\square$

Theorem 3.6 (Whitehead theorem mod $\mathscr{C}$)**.** *Let $\mathscr{C}$ be a Serre class of abelian groups satisfying (ii) and (iii). Let $f\colon X \to Y$ be a map between simply connected spaces and let $n \geq 1$. The following are equivalent:*

1. *$f_*\colon \pi_k(X) \to \pi_k(Y)$ is a $\mathscr{C}$-isomorphism for $k \leq n$ and a $\mathscr{C}$-epimorphism for $k = n+1$.*
2. *$f_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(Y)$ is a $\mathscr{C}$-isomorphism for $k \leq n$ and a $\mathscr{C}$-epimorphism for $k = n+1$.*

Proof. By using the mapping cylinder, we may assume that $f\colon X \to Y$ is an inclusion, without loss of generality. By considering the exact sequences

$$\cdots \to \pi_{k+1}(X) \to \pi_{k+1}(Y) \to \pi_k(Y, X) \to \pi_k(X) \to \cdots,$$

$$\cdots \to \mathrm{H}_{k+1}(X) \to \mathrm{H}_{k+1}(Y) \to \mathrm{H}_k(Y, X) \to \mathrm{H}_k(X) \to \cdots,$$

we see that the first condition is equivalent to $\pi_k(Y, X) \in \mathscr{C}$ for $k \leq n$ and the second to $\mathrm{H}_k(Y, X) \in \mathscr{C}$ for $k \leq n$. There is a relative version of the Hurewicz theorem mod $\mathscr{C}$, which says that $\pi_k(Y, X) \in \mathscr{C}$ for $k \leq n$ if and only if $\mathrm{H}_k(Y, X) \in \mathscr{C}$ for $k \leq n$. The result follows. $\square$

If $\mathscr{C}$ is the class of trivial groups, then the Hurewicz and Whitehead theorems mod $\mathscr{C}$ reduce to the classical theorems.

3.3 Verification of the axioms for certain Serre classes

Proposition 3.7. *Let $\mathscr{C}$ be a Serre class of abelian groups satisfying (ii) and let $F \to E \to B$ be a fibration of path connected spaces, with B simply connected. If two out of three of F, E, B have $\mathrm{H}_k \in \mathscr{C}$ for all $k > 0$, then so does the third.*

Proof. See [11, Lemma 1.9]. □

Corollary 3.8. *Let $\mathscr{C}$ be a Serre class satisfying (ii) and let X be a simply connected space. The following are equivalent:*

1. $\mathrm{H}_k(X)$ *belongs to $\mathscr{C}$ for all $k > 0$.*
2. $\mathrm{H}_k(\Omega X)$ *belongs to $\mathscr{C}$ for all $k > 0$.*

Proof. Apply the proposition to the fibration $\Omega X \to PX \to X$. □

Corollary 3.9. *Let $\mathscr{C}$ be a Serre class satisfying (ii). Then $\mathscr{C}$ satisfies (iii) if and only if the following condition is satisfied:*

(iii)' If A belongs to $\mathscr{C}$, then so does $\mathrm{H}_k(K(A,1))$ for all $k > 0$.

Proof. Use the above corollary and the relation $\Omega K(A,n) \simeq K(A,n-1)$. □

Note that $\mathrm{H}_k(K(A,1))$ is the same as the homology of the group A, so it can be calculated as

$$\mathrm{H}_k(K(A,1)) \cong \mathrm{Tor}_k^{\mathbb{Z}[A]}(\mathbb{Z},\mathbb{Z})$$

where $\mathbb{Z}[A]$ denotes the group ring of A.

Exercise 3.10. Are Proposition 3.7 and its corollaries true if one replaces *Serre class* with *Serre' class*?

Finitely generated groups

Proposition 3.11. *The Serre class of finitely generated abelian groups satisfies (ii) and (iii).*

Proof. First we verify (ii). Let A and B be finitely generated. Then $A \otimes B$ is finitely generated, because if (a_i) and (b_j) generate A and B, respectively, then $(a_i \otimes b_j)$ generate $A \otimes B$. We may choose a presentation $0 \to Q \to P \to A \to 0$ with P and Q finitely generated projective. Then from the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(A,B) \to Q \otimes B \to P \otimes B \to A \otimes B \to 0,$$

it follows that $\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$ is finitely generated, as it is isomorphic to a subgroup of the finitely generated group $Q \otimes B$.

To verify (iii), by Corollary 3.9 it is enough to check that the homology groups $\mathrm{H}_k(K(A,1))$ are finitely generated whenever A is a finitely generated abelian group. By using the Künneth theorem (see e.g., [15, Theorem V.2.1.]), we can reduce to the case when A is a cyclic group. From the calculation of the homology of cyclic groups (see e.g., [15, Chapter VI]) we have

- $\mathrm{H}_k(K(\mathbb{Z},1)) = \mathbb{Z}$ for $k = 0, 1$ and $\mathrm{H}_k(K(\mathbb{Z},1)) = 0$ for $k > 1$.
- $\mathrm{H}_0(K(\mathbb{Z}/n,1) = \mathbb{Z}$, $\mathrm{H}_{2k-1}(K(\mathbb{Z}/n,1)) = \mathbb{Z}/n$ and $\mathrm{H}_{2k}(K(\mathbb{Z}/n,1)) = 0$ for $k > 0$.

In particular, these groups are finitely generated. □

We get the following consequence of Theorem 3.3 and Theorem 3.4.

Corollary 3.12. *The following are equivalent for a simply connected space X:*

1. *The homotopy groups $\pi_n(X)$ are finitely generated for all n.*
2. *The homology groups $\mathrm{H}_n(X)$ are finitely generated for all n.*

In particular, the homotopy groups of a simply connected finite complex are finitely generated.

Torsion groups

Exercise 3.13. Let $\mathscr{C}$ be the Serre class of torsion groups.

1. Prove that $\mathscr{C}$ satisfies (ii) and (iii). (Hint: For (iii), use the fact that for any abelian group G, the homology $\mathrm{H}_k(G;\mathbb{Z})$ is isomorphic to the direct limit $\varinjlim_\alpha \mathrm{H}_k(G_\alpha;\mathbb{Z})$, where $\{G_\alpha\}$ is the directed system of finitely generated subgroups of G.)
2. Prove that a homomorphism $f\colon A \to B$ is a $\mathscr{C}$-isomorphism if and only if the induced homomorphism $f \otimes \mathbb{Q}\colon A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ is an isomorphism.

Uniquely divisible groups

Exercise 3.14. Let $\mathscr{C}$ be the Serre' class of uniquely divisible groups.

1. Use the fact that $A \otimes B$ and $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ are additive functors in each variable to prove that $\mathscr{C}$ satisfies (ii).
2. Prove that if A is a uniquely divisible abelian group, then $\mathrm{H}_k(K(A,1))$ is uniquely divisible for every $k > 0$. Hint: Use that if G is a commutative group, written multiplicatively, then the group homomorphism $(-)^n\colon G \to G$ induces the following map in homology for every k:
$$\begin{aligned} \mathrm{H}_k(G;\mathbb{Z}) &\to \mathrm{H}_k(G;\mathbb{Z}) \\ \alpha &\mapsto n^k\alpha. \end{aligned}$$
3. Prove that $\widetilde{\mathrm{H}}_*(X)$ is uniquely divisible if and only if $\widetilde{\mathrm{H}}_*(X;\mathbb{F}_p) = 0$ for all primes p.
4. Use the Serre spectral sequence with $\mathbb{F}_p$ coefficients to prove that $\widetilde{\mathrm{H}}_*(\Omega X)$ is uniquely divisible if and only if $\widetilde{\mathrm{H}}_*(X)$ is.
5. Conclude that the class of uniquely divisible groups satisfies (iii)

4 Rational homotopy equivalences and localization

4.1 Rational homotopy equivalences

The following is a direct consequence of the Whitehead theorem modulo the class of torsion groups. Note that $\mathrm{H}_n(X;\mathbb{Q}) \cong \mathrm{H}_n(X) \otimes \mathbb{Q}$ since $\mathbb{Q}$ is flat.

Theorem 4.1. *The following are equivalent for a map $f\colon X \to Y$ between simply connected spaces.*

1. *The induced map $f_*\colon \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism.*
2. *The induced map $f_*\colon \mathrm{H}_*(X;\mathbb{Q}) \to \mathrm{H}_*(Y;\mathbb{Q})$ is an isomorphism.*

Definition 4.2. A map $f\colon X \to Y$ between simply connected spaces is called a *rational homotopy equivalence* if the equivalent conditions in Theorem 4.1 are satisfied. We shall write

$$f\colon X \xrightarrow{\sim_{\mathbb{Q}}} Y$$

to indicate that f is a rational homotopy equivalence.

Definition 4.3. We say that two simply connected spaces X and Y have the same *rational homotopy type*, or are *rationally homotopy equivalent*, if there is a zig-zag of rational homotopy equivalences connecting X and Y;

$$X \xleftarrow{\sim_{\mathbb{Q}}} Z_1 \xrightarrow{\sim_{\mathbb{Q}}} \cdots \xleftarrow{\sim_{\mathbb{Q}}} Z_n \xrightarrow{\sim_{\mathbb{Q}}} Y.$$

We will write $X \sim_{\mathbb{Q}} Y$ to indicate that X and Y have the same rational homotopy type.

4.2 $\mathbb{Q}$-localization

Standard references for localizations are [26, 14]. Localizations are also discussed in [11].

Theorem 4.4. *The following are equivalent for a simply connected space Y.*

1. *The homotopy groups $\pi_k(Y)$ are uniquely divisible for all k.*
2. *The homology groups $\mathrm{H}_k(Y)$ are uniquely divisible for all $k > 0$.*

Proof. The class of uniquely divisible groups is a Serre' class by Exercise 3.2, and it satisfies (ii) and (iii) by Exercise 3.14, so the equivalence of the first two conditions follows from Theorem 3.4. □

Definition 4.5. A simply connected space Y is called *rational*, or $\mathbb{Q}$-*local*, if the equivalent conditions in Theorem 4.4 are satisfied.

Definition 4.6. A *rationalization*, or $\mathbb{Q}$-*localization*, of a simply connected space X is a $\mathbb{Q}$-local space $X_{\mathbb{Q}}$ together with a rational homotopy equivalence $r\colon X \to X_{\mathbb{Q}}$. In other words, the induced maps

$$r_*\colon \mathrm{H}_k(X;\mathbb{Q}) \to \mathrm{H}_k(X_{\mathbb{Q}}),$$
$$r_*\colon \pi_k(X) \otimes \mathbb{Q} \to \pi_k(X_{\mathbb{Q}}),$$

are isomorphisms for all $k > 0$.

Theorem 4.7. *The following are equivalent for a simply connected space Y*

1. *Y is $\mathbb{Q}$-local.*
2. *For every rational homotopy equivalence $r\colon Z \to X$ between CW-complexes, the induced map*
$$r^*\colon [X,Y] \to [Z,Y]$$
is a bijection.

We will prove this in three steps:

- Prove $\Rightarrow$.
- Prove the existence of $\mathbb{Q}$-localizations.
- Prove $\Leftarrow$.

Lemma 4.8. *A map $f\colon X \to Y$ between simply connected spaces is a rational homotopy equivalence if and only if*
$$f^*\colon \mathrm{H}^n(Y;A) \to \mathrm{H}^n(X;A)$$
is an isomorphism for all uniquely divisible abelian groups A and all n.

Proof. By the universal coefficient theorem (see [15, Theorem V.3.3]), there is a short exact sequence
$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-1}(X), A) \to \mathrm{H}^n(X;A) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(X), A) \to 0,$$
which is natural in X. If A is divisible, then A is injective as a $\mathbb{Z}$-module, so $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-1}(X), A) = 0$. If A is uniquely divisible, then it admits a unique $\mathbb{Q}$-vector space structure, and there results a natural isomorphism
$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(X), A) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{H}_n(X) \otimes \mathbb{Q}, A).$$
Noting that $\mathrm{H}_n(X;\mathbb{Q}) \cong \mathrm{H}_n(X) \otimes \mathbb{Q}$, we get the following commutative diagram
$$\begin{tikzcd}
\mathrm{H}^n(Y;A) \arrow[r, "\cong"] \arrow[d, "f^*"] & \mathrm{Hom}_{\mathbb{Q}}(\mathrm{H}_n(Y;\mathbb{Q}), A) \arrow[d, "f^*"] \\
\mathrm{H}^n(Z;A) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Q}}(\mathrm{H}_n(Z;\mathbb{Q}), A),
\end{tikzcd}$$
which proves the claim. ◻

Proof of $\Rightarrow$ in Theorem 4.7. For Y an Eilenberg-Mac Lane space $K(A,n)$, with A uniquely divisible, one argues using the commutative diagram
$$\begin{tikzcd}
{[X, K(A,n)]} \arrow[r, "r^*"] \arrow[d, "\cong"'] & {[Z, K(A,n)]} \arrow[d, "\cong"'] \\
\mathrm{H}^n(X;A) \arrow[r, "r^*"] & \mathrm{H}^n(Z;A).
\end{tikzcd}$$
The lower horizontal map is an isomorphism by Lemma 4.8.

Next, let Y be an arbitrary simply connected $\mathbb{Q}$-local space and consider

$$r^*\colon [Z,Y] \to [X,Y]$$

for a rational homotopy equivalence $r\colon X \to Z$ between CW-complexes.

<u>r^* is surjective:</u> Let $f\colon X \to Y$ be a given map. We have to solve the lifting problem

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"'] & Y \\ Z \arrow[ur, dashed, "\lambda"'] & \end{tikzcd}$$

We do this by crawling up the Postnikov tower of Y. Consider the composite $f_n\colon X \to Y \to Y_n$ where Y_n is the n^{th} Postnikov section of Y. Suppose by induction that $f_k\colon X \to Y_k$ admits an extension $\lambda_k\colon Z \to Y_k$ for $k < n$. Then we need to solve the following lifting problem.

$$\begin{tikzcd} X \arrow[r, "f_n"] \arrow[d, "r"'] & Y_n \arrow[d] \\ Z \arrow[ur, dashed, "\lambda_n"] \arrow[r, "\lambda_{n-1}"'] & Y_{n-1}. \end{tikzcd}$$

Now, $Y_n \to Y_{n-1}$ is a pullback of the path fibration over $K(\pi_n(Y), n+1)$ along the k-invariant $k^{n+1}\colon Y_{n-1} \to K(\pi_n(Y), n+1)$, see (1). Therefore, the above lifting problem is equivalent to the following:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "r"'] & * \arrow[d] \\ Z \arrow[ur, dashed] \arrow[r, "x"'] & K(\pi_n(Y), n+1), \end{tikzcd}$$

where $*$ is short for the contractible path space, and $x = k^{n+1} \circ \lambda_{n-1}$. But this can be reformulated as follows: "Given $x \in \mathrm{H}^{n+1}(Z; \pi_n(Y))$ such that $r^*(x) = 0 \in \mathrm{H}^{n+1}(X; \pi_n(Y))$ is it true that $x = 0$?". This is indeed true because $r^*\colon \mathrm{H}^{n+1}(Z; \pi_n(Y)) \to \mathrm{H}^{n+1}(X; \pi_n(Y))$ is an isomorphism, as $\pi_n(Y)$ is uniquely divisible and r is a rational homotopy equivalence. Before we claim to have finished, we should point out that the induction starts with λ_1 the trivial map from Z to $Y_1 = *$

<u>r^* is injective:</u> Given two lifts

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"'] & Y \\ Z \arrow[ur, shift left, "\lambda_0"] \arrow[ur, shift right, "\lambda_1"'] & \end{tikzcd}$$

we will prove that λ_0 is homotopic to λ_1 relative to X. Finding a homotopy $h\colon Z \times I \to Y$ between λ_0 and λ_1 relative to X is equivalent to solving the lifting problem

$$\begin{tikzcd} X \times I \bigcup_{X \times \partial I} Z \times \partial I \arrow[r] \arrow[d] & Y \\ Z \times I \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

where the top map is defined on the respective summands by

$$X \times I \overset{proj.}{\to} X \overset{f}{\to} Y, \quad (\lambda_0, \lambda_1)\colon Z \times \partial I \to Y.$$

To solve this lifting problem, we note that the vertical map is a rational homotopy equivalence because of the following calculation of relative homology groups:

$$\mathrm{H}_n(Z \times I, X \times I \cup_{X \times \partial I} Z \times \partial I; \mathbb{Q}) \cong \mathrm{H}_{n-1}(Z, Y; \mathbb{Q}) = 0.$$

Then the argument proceeds as before. □

Theorem 4.9. *Every simply connected space X admits a $\mathbb{Q}$-localization.*

Proof. We will construct $X_{\mathbb{Q}}$ by induction on a Postnikov tower of X. First, for an Eilenberg-Mac Lane space $K(A, n)$, the obvious map

$$r\colon K(A, n) \to K(A \otimes \mathbb{Q}, n)$$

is a rationalization. In particular, since $X_2 = K(\pi_2, 2)$, we can start the induction by taking $(X_2)_{\mathbb{Q}} = K(\pi_2 \otimes \mathbb{Q}, 2)$. Suppose inductively that we have constructed rationalizations $r_k\colon X_k \to (X_k)_{\mathbb{Q}}$ for $k < n$. The space X_n is the pullback of the path space over $K(\pi_n, n+1)$ along the k-invariant $k^{n+1}\colon X_{n-1} \to K(\pi_n, n+1)$, i.e., the back square in the diagram below is a pullback.

$$\begin{tikzcd}
X_n \arrow[rr] \arrow[d] & & PK(\pi_n, n+1) \arrow[d] & \\
X_{n-1} \arrow[rr, "k^{n+1}"] \arrow[dr, "r_{n-1}"'] & & K(\pi_n, n+1) \arrow[dr, "r"] & \\
& (X_{n-1})_{\mathbb{Q}} \arrow[rr, dashed, "(k^{n+1})_{\mathbb{Q}}"] & & K(\pi_n \otimes \mathbb{Q}, n+1)
\end{tikzcd}$$

The diagonal arrows are rationalizations, and the dotted horizontal arrow is the preimage of $r \circ k^{n+1}$ under the map

$$r_{n-1}^*\colon [(X_{n-1})_{\mathbb{Q}}, K(\pi_n \otimes \mathbb{Q}, n+1)] \to [X_{n-1}, K(\pi_n \otimes \mathbb{Q}, n+1)],$$

which is a bijection by the $\Rightarrow$ part of the theorem. If we define $(X_n)_{\mathbb{Q}}$ to be the pullback of the path space over $K(\pi_n \otimes \mathbb{Q}, n+1)$ along $(k^{n+1})_{\mathbb{Q}}$, then there is an induced map $r_n\colon X_n \to (X_n)_{\mathbb{Q}}$. By looking at the long exact sequence of homotopy groups, it follows that r_n is a rationalization. There results a tower of fibrations

$$\cdots \to (X_n)_{\mathbb{Q}} \to (X_{n-1})_{\mathbb{Q}} \to \cdots \to (X_2)_{\mathbb{Q}}.$$

The inverse limit $\varprojlim (X_n)_{\mathbb{Q}}$ is a rationalization of X. □

Proof of $\Leftarrow$ in Theorem 4.7. Theorem 4.9 asserts the existence of a $\mathbb{Q}$-localization $Y \to Y_{\mathbb{Q}}$. We may assume that $Y_{\mathbb{Q}}$ is a CW-complex. The map $r\colon Y \to Y_{\mathbb{Q}}$ is a rational homotopy equivalence, so the induced map $r^*\colon [Y_{\mathbb{Q}}, Y] \to [Y, Y]$ is a bijection. In particular, we can find $s\colon Y_{\mathbb{Q}} \to Y$ such that $s \circ r \simeq 1_Y$. We must also have $r \circ s \simeq 1_{Y_{\mathbb{Q}}}$, because both map to r under the map $r^*\colon [Y_{\mathbb{Q}}, Y_{\mathbb{Q}}] \to [Y, Y_{\mathbb{Q}}]$, which is a bijection since $Y_{\mathbb{Q}}$ is $\mathbb{Q}$-local. Hence, Y and $Y_{\mathbb{Q}}$ are homotopy equivalent. In particular, they have isomorphic homology and homotopy groups, which implies that Y is $\mathbb{Q}$-local as well. □

We end this section by noting some easy consequences.

Theorem 4.10.

- *Every rational homotopy equivalence between $\mathbb{Q}$-local CW-complexes is a homotopy equivalence.*
- *Any two $\mathbb{Q}$-localizations of X are homotopy equivalent relative to X.*

Once we have the characterization in Theorem 4.7, the proof of this is entirely formal. See Theorem 4.14 below.

4.3 Appendix: Abstract localization

The notion of $\mathbb{Q}$-localization fits into a general scheme. Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a class of morphisms in $\mathcal{C}$. For objects X and Y of $\mathcal{C}$, write $\mathcal{C}(X, Y)$ for the set of morphisms in $\mathcal{C}$ from X to Y.

Definition 4.11.

- An object X in $\mathcal{C}$ is called *$\mathcal{W}$-local* if
$$f^*\colon \mathcal{C}(B, X) \to \mathcal{C}(A, X)$$
is a bijection for all morphisms $f\colon A \to B$ that belong to $\mathcal{W}$.
- A *$\mathcal{W}$-localization* of an object X is an object $X_{\mathcal{W}}$ together with a morphism
$$\ell\colon X \to X_{\mathcal{W}},$$
such that
  - The object $X_{\mathcal{W}}$ is $\mathcal{W}$-local.
  - The morphism ℓ belongs to $\mathcal{W}$.

Proposition 4.12 (Universal property of localization)**.** *A $\mathcal{W}$-localization $\ell\colon X \to X_{\mathcal{W}}$, if it exists, satisfies the following universal properties.*

1. *The object $X_{\mathcal{W}}$ is the initial $\mathcal{W}$-local object under X, i.e., any map $f\colon X \to Y$ from X to a $\mathcal{W}$-local object Y factors uniquely as*
$$\begin{tikzcd} X \arrow[r, "\ell"] \arrow[dr, "f"'] & X_{\mathcal{W}} \arrow[d, dashed, "\exists!"] \\ & Y \end{tikzcd}$$
2. *The morphism ℓ is the terminal morphism in $\mathcal{W}$ out of X, i.e., for any morphism $w\colon X \to Z$ in $\mathcal{W}$, there is a unique factorization as*
$$\begin{tikzcd} X \arrow[r, "\ell"] \arrow[d, "w"'] & X_{\mathcal{W}} \\ Z \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proof. The first statement is simply spelling out that $\ell^*\colon \mathcal{C}(X_{\mathcal{W}}, Y) \to \mathcal{C}(X, Y)$ is a bijection, which holds because Y is $\mathcal{W}$-local and ℓ belongs to $\mathcal{W}$. The second statement is spelling out that $w^*\colon \mathcal{C}(Z, X_{\mathcal{W}}) \to \mathcal{C}(X, X_{\mathcal{W}})$ is a bijection, which holds since $X_{\mathcal{W}}$ is $\mathcal{W}$-local and w belongs to $\mathcal{W}$. □

Exercise 4.13. Let $\mathcal{C}$ be the category of abelian groups, and let $\mathcal{W}$ be the class of $\mathscr{C}$-isomorphisms, where $\mathscr{C}$ is the class of torsion groups. Show that

1. An abelian group A is $\mathcal{W}$-local precisely when it is uniquely divisible.
2. For every abelian group A, the canonical map $A \to A \otimes \mathbb{Q}$ is a $\mathcal{W}$-localization.

Theorem 4.14. *1. Every map in $\mathcal{W}$ between $\mathcal{W}$-local objects is an isomorphism.*

2. Any two $\mathcal{W}$-localizations of X are canonically isomorphic under X.

Proof. Let $w\colon X \to Y$ be a morphism in $\mathcal{W}$ between $\mathcal{W}$-local objects. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{C}(Y,X) \arrow[r, "\cong", "w^*"'] \arrow[d, "w_*"] & \mathcal{C}(X,X) \arrow[d, "w_*"] \\ \mathcal{C}(Y,Y) \arrow[r, "\cong", "w^*"'] & \mathcal{C}(X,Y). \end{tikzcd}$$

Let $s \in \mathcal{C}(Y, X)$ be the preimage of 1_X, so that $1_X = w^*(s) = s \circ w$. Then under the bijection $w^*\colon \mathcal{C}(Y, Y) \to \mathcal{C}(X, Y)$, both 1_Y and $w \circ s$ are mapped to w. Hence $w \circ s = 1_Y$. Thus, w is an isomorphism with inverse s. We leave the proof of the second statement to the reader. □

To see how $\mathbb{Q}$-localizations discussed in the previous section fit into the abstract framework, take $\mathcal{C}$ to be the homotopy category of simply connected spaces, i.e., the category of all simply connected spaces and continuous maps with the weak homotopy equivalences formally inverted. This category is equivalent to the category whose objects are all simply connected CW-complexes, and where the set of morphisms from X to Y is the set $[X, Y]$ of homotopy classes of continuous maps from X to Y. We take $\mathcal{W}$ to be the class of rational homotopy equivalences. Theorem 4.7 says that an object in $\mathcal{C}$ is $\mathbb{Q}$-local, in the sense of Definition 4.5, if and only if it is $\mathcal{W}$-local in the sense of Definition 4.11.

Remark 4.15. Let P be a set of prime numbers. Everything we did in in this section works equally well if one makes the following replacements.

- $\mathbb{Q}$ is replaced by $\mathbb{Z}[P^{-1}]$.
- 'Torsion group' is replaced by 'torsion group with torsion relatively prime to all $p \notin P$.'
- 'Uniquely divisible' is replaced by 'uniquely p-divisible for all $p \in P$'.

If P is the set of all primes then we of course recover the rational setting. If P is the set of all primes except a given one p, then it is common to denote $\mathbb{Z}[P^{-1}]$ by $\mathbb{Z}_{(p)}$.

5 Rational cohomology of Eilenberg-Mac Lane spaces and rational homotopy groups of spheres

5.1 Rational cohomology of Eilenberg-Mac Lane spaces

Theorem 5.1. *The rational cohomology algebra of an Eilenberg-Mac Lane space* $\mathrm{H}^*(K(\mathbb{Z},n);\mathbb{Q})$ *is a free graded commutative algebra on the class* x*, i.e., a polynomial algebra* $\mathbb{Q}[x]$ *if* n *is even and an exterior algebra* $E(x)$ *if* n *is odd.*

Proof. The proof is by induction on n. For $n = 1$ we have that $K(\mathbb{Z},1) = S^1$ and clearly $\mathrm{H}^*(S^1;\mathbb{Q})$ is an exterior algebra on a generator of degree 1. For even $n \geq 2$, consider the path-loop fibration

$$\Omega K(\mathbb{Z},n) \to PK(\mathbb{Z},n) \to K(\mathbb{Z},n).$$

By induction, the rational cohomology of $\Omega K(\mathbb{Z},n) \simeq K(\mathbb{Z},n-1)$ is an exterior algebra on a generator x of degree $n-1$. The rational Serre spectral sequence of the fibration has E^2-term

$$E_2^{p,q} = \mathrm{H}^p(K(\mathbb{Z},n);\mathbb{Q}) \otimes \mathrm{H}^q(K(\mathbb{Z},n-1);\mathbb{Q}),$$

and since $PK(\mathbb{Z},n)$ is contractible, $E_\infty^{p,q} = 0$ for $(p,q) \neq (0,0)$. Since $E_2^{p,q} = 0$ for $q \neq 0, n-1$, the only possible non-zero differential is $d_n\colon E_n^{p,n-1} \to E_n^{p+n,0}$. Hence, $E_2 = E_3 = \cdots = E_n$, and for every p there is an exact sequence

$$0 \longrightarrow E_\infty^{p,n-1} \longrightarrow E_n^{p,n-1} \xrightarrow{d_n} E_n^{p+n,0} \longrightarrow E_\infty^{p+n,0} \longrightarrow 0$$

Since $E_\infty^{p,q} = 0$ for $(p,q) \neq (0,0)$, this shows that d_n is an isomorphism. Consider $x \in E_n^{n-1,0}$ and let $y = d_n(x) \in E_n^{0,n} = \mathrm{H}^n(K(\mathbb{Z},n);\mathbb{Q})$. Then xy generates $E_n^{n,n-1}$, and since d_n is a derivation, $d_n(xy) = d_n(x)y = y^2$ generates $E_n^{2n,0}$. By induction, xy^k generates $E_n^{kn,n-1}$ whence $d_n(xy^k) = y^{k+1}$ generates $\mathrm{H}^{(k+1)n}(K(\mathbb{Z},n);\mathbb{Q})$. Thus, $\mathrm{H}^*(K(\mathbb{Z},n);\mathbb{Q})$ is a polynomial algebra on y.

The proof for odd n is left to the reader as an exercise. □

5.2 Rational homotopy groups of spheres

Theorem 5.2. *If* n *is odd, then* $\pi_k(S^n)$ *is finite for* $k \neq n$.

Proof. Let $f\colon S^n \to K(\mathbb{Z},n)$ represent a generator of

$$\pi_n(K(\mathbb{Z},n)) \cong [S^n, K(\mathbb{Z},n)] \cong \mathrm{H}^n(S^n) \cong \mathbb{Z}.$$

It follows from the Hurewicz theorem that the induced map in cohomology

$$f^*\colon\ \mathrm{H}^*(K(\mathbb{Z},n);\mathbb{Q}) \to \mathrm{H}^*(S^n;\mathbb{Q}) \tag{2}$$

is an isomorphism in dimension $* = n$. If n is odd, then by the calculation in the previous section, both sides of (2) are exterior algebras on a single generator in degree n, so it follows that (2) is an isomorphism. Hence, f is a rational homotopy equivalence, and Theorem 4.1 implies that

$$f_*\colon \pi_*(S^n) \otimes \mathbb{Q} \to \pi_*(K(\mathbb{Z},n)) \otimes \mathbb{Q}$$

is an isomorphism. Hence, $\pi_n(S^n) \otimes \mathbb{Q} = \mathbb{Q}$ and $\pi_k(S^n) \otimes \mathbb{Q} = 0$ for $k \neq n$. Since $\pi_k(S^n)$ is finitely generated, this implies that $\pi_k(S^n)$ is finite for $k \neq n$. □

Theorem 5.3. *If n is even, then $\pi_k(S^n)$ is finite for $k \neq n$, $2n-1$. The group $\pi_{2n-1}(S^n)$ has rank one.*

Proof. Let x denote a generator of $\mathrm{H}^n(K(\mathbb{Z},n)) \cong \mathbb{Z}$, and let F be the homotopy fiber of a map $K(\mathbb{Z},n) \to K(\mathbb{Z},2n)$ that represents the class

$$x^2 \in \mathrm{H}^{2n}(K(\mathbb{Z},n)) = [K(\mathbb{Z},n), K(\mathbb{Z},2n)].$$

As before, let $f\colon S^n \to K(\mathbb{Z},n)$ represent a generator. Since $\pi_n K(\mathbb{Z},2n) = 0$, it follows that f factors through the homotopy fiber

$$\begin{tikzcd} F \arrow[r] & K(\mathbb{Z},n) \arrow[r, "x^2"] & K(\mathbb{Z},2n) \\ & S^n \arrow[ul, dashed, "g"] \arrow[u, "f"'] \arrow[ur, "\simeq *"'] & \end{tikzcd}$$

¿From the long exact sequence in homotopy

$$\cdots \to \pi_{k+1}(K(\mathbb{Z},2n)) \to \pi_k(F) \to \pi_k(K(\mathbb{Z},n)) \to \pi_k(K(\mathbb{Z},2n)) \to \cdots$$

one deduces that $\pi_k(F) = 0$ for $k \neq n$, $2n-1$, and $\pi_n(F) \cong \pi_{2n-1}(F) \cong \mathbb{Z}$. Moreover, $g_*\colon \pi_n(S^n) \to \pi_n(F)$ is an isomorphism, so by the Hurewicz theorem, the map $g^*\colon \mathrm{H}^*(F;\mathbb{Q}) \to \mathrm{H}^*(S^n;\mathbb{Q})$ is an isomorphism in degree n. If we can prove that $\mathrm{H}^*(F;\mathbb{Q})$ is an exterior algebra on the generator in degree n, then it follows that g is a rational homotopy equivalence, and we are done. $\square$

Exercise 5.4. Let F denote the homotopy fiber of the map $x^2\colon K(\mathbb{Z},n) \to K(\mathbb{Z},2n)$. Prove that $\mathrm{H}^*(F;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)$ where x is a generator of degree n.

We get different answers for the cohomology algebra $\mathrm{H}^*(K(\mathbb{Z},n);\mathbb{Q})$ depending on whether n is even or odd. But these can be summarized by saying that $\mathrm{H}^*(K(\mathbb{Z},n);\mathbb{Q})$ is a free graded commutative algebra on a generator of degree n. We also get different answers for what $\pi_*(S^n) \otimes \mathbb{Q}$ is depending on whether n is even or odd. Is there a way of summarizing these? The answer is yes, and to formulate this properly we need to introduce Whitehead products.

5.3 Whitehead products

The Whitehead product is a certain operation on the homotopy groups of a space X,

$$[-,-]\colon \pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X). \tag{3}$$

It is defined by sending a pair of homotopy classes of maps

$$S^p \xrightarrow{\alpha} X, \quad S^q \xrightarrow{\beta} X,$$

to the homotopy class of the composite

$$S^{p+q-1} \xrightarrow{w_{p,q}} S^p \vee S^q \xrightarrow{\alpha\vee\beta} X.$$

$$[\alpha,\beta] = (\alpha \vee \beta) \circ w_{p,q},$$

where the map

$$w_{p,q}\colon S^{p+q-1} \to S^p \vee S^q,$$

called the *universal Whitehead product*, is constructed as follows. Represent the sphere as a disk modulo its boundary; $S^p = D^p/\partial D^p$. Consider the diagram

$$\begin{tikzcd}
D^p \times D^q \arrow[rr] & & S^p \times S^q \\
S^{p+q-1} \cong \partial(D^p \times D^q) \arrow[u] \arrow[rr, dashed, "w_{p,q}"] & & S^p \vee S^q \arrow[u]
\end{tikzcd} \tag{4}$$

where the top map collapses the boundary of each disk factor to a point. The dotted arrow exists, because the image of the boundary

$$\partial(D^p \times D^q) = \partial D^p \times D^q \cup D^p \times \partial D^q$$

under the collapsing map $D^p \times D^q \to S^p \times S^q$ is contained in the wedge

$$S^p \vee S^q = * \times S^q \cup S^p \times * \subset S^p \times S^q.$$

Alternative definition. If the classes $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ are represented by maps of pairs

$$f\colon (D^p, \partial D^p) \to (X, *), \quad g\colon (D^q, \partial D^q) \to (X, *),$$

then the class $[\alpha, \beta] \in \pi_{p+q-1}(X)$ is represented by the map

$$S^{p+q-1} \cong \partial(D^p \times D^q) = \partial D^p \times D^q \cup D^p \times \partial D^q \xrightarrow{h} X,$$

where

$$h(x, y) = \begin{cases} f(x), & x \in D^p,\ y \in \partial D^q \\ g(y), & x \in \partial D^p,\ y \in D^q \end{cases}$$

Exercise 5.5. Show that for $p = q = 1$, the Whitehead product $\pi_1(X) \times \pi_1(X) \to \pi_1(X)$ agrees with the commutator operation $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ in the fundamental group.

Properties of the Whitehead product. The Whitehead product satisfies the following identities. Let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$ and $\gamma \in \pi_r(X)$, and assume that $p, q, r \geq 2$. Then

1. $[-,-]\colon \pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X)$ is bilinear if $p, q \geq 2$.
2. $[\alpha, \beta] = (-1)^{pq}[\beta, \alpha]$.
3. $(-1)^{pr}[[\alpha, \beta], \gamma] + (-1)^{qp}[[\beta, \gamma], \alpha] + (-1)^{rq}[[\gamma, \alpha], \beta] = 0$.

If X is an H-space, for instance a loop space, then the Whitehead product is trivial in $\pi_*(X)$. This generalizes the fact that the fundamental group $\pi_1(X)$ is abelian if X is an H-space.

The Whitehead product is an unstable operation: if $E\colon \pi_{p+q-1}(X) \to \pi_{p+q}(\Sigma X)$ denotes the suspension homomorphism, then

$$E[\alpha, \beta] = 0$$

for all $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$.

The Hopf invariant. Let n be an even positive integer. Consider a map $f\colon S^{2n-1} \to S^n$. Form the adjunction space X_f by attaching a $2n$-cell along f, i.e., form a pushout

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f"] \arrow[d] & S^n \arrow[d] \\ D^{2n} \arrow[r] & X_f. \end{tikzcd}$$

The space X_f is a CW-complex with one cell each in dimensions 0, n and $2n$. Therefore, the integral cohomology has the form

$$\mathrm{H}^*(X_f) = \mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}y$$

where x and y are generators of dimension n and $2n$, respectively. We have that

$$x^2 = H(f)y$$

for a certain integer $H(f)$, and the cohomology ring is completely determined by $H(f)$. The integer $H(f)$ is called the *Hopf invariant* of f.

It is a fact that the Hopf invariant defines a homomorphism

$$H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}.$$

In particular, if $H(f) \neq 0$ then $[f] \in \pi_{2n-1}(S^n)$ is not torsion, so $[f]$ will be a generator of the rational homotopy group $\pi_{2n-1}(S^n) \otimes \mathbb{Q} \cong \mathbb{Q}$.

Exercise 5.6. Let $\iota\colon S^n \to S^n$ be the identity map of an even dimensional sphere. In this exercise, we will prove that the Whitehead product

$$[\iota, \iota]\colon S^{2n-1} \to S^n$$

has Hopf invariant ± 2.

1. Let $A \subset X$ and $B \subset Y$ be subspaces. Prove that there is a pushout diagram

$$\begin{tikzcd} X \times B \bigcup_{A \times B} A \times Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ X/A \vee Y/B \arrow[r] & X/A \times Y/B. \end{tikzcd}$$

   Conclude that the diagram (4) is a pushout diagram. (Thus, the universal Whitehead product $w_{p,q}\colon S^{p+q-1} \to S^p \vee S^q$ is an attaching map for the $(p+q)$-cell of $S^p \times S^q$.)

2. Argue that there is a diagram

$$\begin{tikzcd} S^{2n-1} \arrow[r, "w_{n,n}"] \arrow[d] & S^n \vee S^n \arrow[r, "\nabla"] \arrow[d] & S^n \arrow[d] \\ D^n \times D^n \arrow[r] & S^n \times S^n \arrow[r, "g"] & X_{[\iota,\iota]} \end{tikzcd} \tag{5}$$

   where both squares are pushouts.

3. The cohomology ring of $S^n \times S^n$ is given by

$$\mathrm{H}^*(S^n \times S^n) = \mathbb{Z} \oplus \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}ab,$$

where a and b are generator of degree n. Use the diagram (5) to argue that the map

$$g^*\colon\ \mathrm{H}^*(X_{[\iota,\iota]}) \to \mathrm{H}^*(S^n \times S^n)$$

sends the generator x to $a+b$ and that it is an isomorphism in degree $2n$. Use this fact to prove that $x^2 = 2y$ in the cohomology ring of $X_{[\iota,\iota]}$.

Remark 5.7. The Whitehead product $[\iota,\iota]$ has Hopf invariant 2 and for $n \neq 2,4,8$ the $\mathbb{Z}$-summand in $\pi_{2n-1}(S^n)$ is generated by $[\iota,\iota]$. The $\mathbb{Z}$-summands of $\pi_3(S^2)$, $\pi_7(S^4)$ and $\pi_{15}(S^8)$ are generated by the Hopf maps η. The relation $2\eta = [\iota,\iota]$ holds in these dimensions.

Whitehead algebras and graded Lie algebras.

Definition 5.8. A *Whitehead algebra* is a graded $\mathbb{Q}$-vector space π_* together with a bilinear bracket

$$[-,-]\colon \pi_p \otimes \pi_q \to \pi_{p+q-1},$$

satisfying the following axioms: for $\alpha \in \pi_p$, $\beta \in \pi_q$ and $\gamma \in \pi_r$,

1. (Symmetry) $[\alpha,\beta] = (-1)^{pq}[\beta,\alpha]$.
2. (Jacobi relation) $(-1)^{pr}[[\alpha,\beta],\gamma] + (-1)^{qp}[[\beta,\gamma],\alpha] + (-1)^{rq}[[\gamma,\alpha],\beta] = 0$.

If X is a simply connected space, then the rational homotopy groups $\pi_*(X)\otimes\mathbb{Q}$ together with the Whitehead product is a Whitehead algebra.

Let us examine the free Whitehead algebra $\mathbb{W}[\alpha]$ on a generator α of degree n. If n is odd, then the symmetry condition says that

$$[\alpha,\alpha] = -[\alpha,\alpha],$$

which implies that $[\alpha,\alpha] = 0$. Hence,

$$\mathbb{W}[\alpha] = \mathbb{Q}\alpha$$

in this case. If n is even, then $[\alpha,\alpha] \neq 0$ in the free Whitehead algebra $\mathbb{W}[\alpha]$, but it is a consequence of the Jacobi relation that $3[[\alpha,\alpha],\alpha] = 0$. Thus,

$$\mathbb{W}[\alpha] = \mathbb{Q}\alpha \oplus \mathbb{Q}[\alpha,\alpha]$$

where $[\alpha,\alpha]$ has degree $2n-1$. Thus, the Whitehead product allows us to summarize our calculations of the rational homotopy groups of spheres as follows.

Theorem 5.9. *The rational homotopy groups $\pi_*(S^n)\otimes\mathbb{Q}$ is a free graded Whitehead algebra generated by the class of the identity map ι.*

Definition 5.10. A *graded Lie algebra* is a graded $\mathbb{Q}$-vector space L_* together with a binary operation

$$[-,-]\colon L_p \otimes L_q \to L_{p+q}$$

such that for all $x \in L_p$, $y \in L_q$ and $z \in L_r$, the following relations hold.

1. (Anti-symmetry) $[x, y] = -(-1)^{pq}[y, x]$.
2. (Jacobi relation) $(-1)^{pr}[[x, y], z] + (-1)^{qp}[[y, z], x] + (-1)^{rq}[[z, x], y] = 0$.

Exercise 5.11. Let L_* be a graded Lie algebra. Set $\pi_* = sL_*$, so that $\pi_{p+1} = L_p$, and define a bracket $\pi_{p+1} \otimes \pi_{q+1} \to \pi_{p+q+1}$ by the formula

$$[sx, sy] := (-1)^p s[x, y]$$

for $x \in L_p$ and $y \in L_q$. Prove that this makes π_* into a Whitehead algebra.

6 Interlude: Simplicial objects

In this section, we will recall some basic facts from simplicial homotopy theory. Proofs may be found in [8, 17].

6.1 Simplicial objects

Let Δ be the category whose objects are the ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$, and whose morphisms are all non-decreasing functions between these, i.e., functions $\varphi\colon [m] \to [n]$ such that $\varphi(i) \leq \varphi(i + 1)$ for all $i \in [m]$.

Definition 6.1. A *simplicial object* in category $\mathcal{C}$ is a functor

$$X\colon \Delta^{op} \to \mathcal{C}.$$

A morphism between simplicial objects is a natural transformation.

The category of simplicial objects in $\mathcal{C}$ is denoted $\mathcal{C}^{\Delta^{op}}$ or $s\mathcal{C}$. It is common to use the notation $X_n = X([n])$ and $\varphi^* = X(\varphi)$.

The category Δ is generated by the maps (for $n \geq 0$)

$$d^i\colon [n-1] \to [n], \quad 0 \leq i \leq n,$$

$$s^i\colon [n+1] \to [n], \quad 0 \leq i \leq n,$$

where d^i is the unique non-decreasing injective function whose image does not contain i, and s^i is the unique non-decreasing surjective function with $s^i(i) = s^i(i + 1) = i$. These maps satisfy the following relations

$$\begin{aligned}
d^j d^i &= d^i d^{j-1}, && i < j \\
s^j d^i &= d^i s^{j-1}, && i < j \\
s^i d^i &= s^i d^{i+1} = 1 && \\
s^j d^i &= d^{i-1} s^j, && i > j + 1 \\
s^j s^i &= s^i s^{j+1}, && i \leq j.
\end{aligned}$$

This list of relations is complete in the sense that every relation that holds between composites of s^i's and d^i's can be derived from it. Furthermore, the s^i's and the d^i's are generators for the category Δ in the sense that any morphism in Δ can be written as a composite of s^i's and d^i's. As a consequence, there is an alternative definition of a simplicial object X as a sequence of objects $X_0, X_1, \ldots$, in $\mathcal{C}$ together with maps for all $n \geq 0$,

$$d_i \colon X_n \to X_{n-1}, \quad 0 \leq i \leq n,$$

$$s_i \colon X_n \to X_{n+1}, \quad 0 \leq i \leq n,$$

satisfying the *simplicial identities*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j \\ d_i s_j &= s_{j-1} d_i, & i < j \\ d_i s_i &= d_{i+1} s_i = 1 & \\ d_i s_j &= s_j d_{i-1}, & i > j+1 \\ s_i s_j &= s_{j+1} s_i, & i \leq j. \end{aligned}$$

The map d_i is called the 'i^{th} face map' and s_i is called the 'i^{th} degeneracy map'.

Elements of the set X_n are called *n-simplices of X*. For an n-simplex x, and a morphism $\varphi \colon [m] \to [n]$, we will sometimes write

$$x(\varphi(0), \ldots, \varphi(m)), \quad \text{or} \quad x_{\varphi(0) \ldots \varphi(m)},$$

for the m-simplex $\varphi^*(x)$. For example, if $x \in X_2$, then $x_{012} = x$, $x_{02} = d_1(x)$, $x_0 = d_1 d_2(x)$ etc.

6.2 Examples of simplicial sets

The singular simplicial set

The singular simplicial set, or singular complex, of a topological space T is the simplicial set $S_\bullet(T)$ whose set of n-simplices is the set

$$S_n(T) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, T)$$

of all continuous maps $f \colon \Delta^n \to T$ from the standard topological n-simplex

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \;\; t_0 + \ldots + t_n = 1 \right\}.$$

For a morphism $\varphi \colon [m] \to [n]$ in Δ, we have a continuous map

$$\varphi_* \colon \Delta^m \to \Delta^n, \quad \varphi_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n),$$

defined by

$$s_i = \sum_{j \in \varphi^{-1}(j)} t_j.$$

The structure map $\varphi^* \colon S_n(T) \to S_m(T)$ is defined by $\varphi^*(f) = f \circ \varphi_*$.

Simplicial sets from abstract simplicial complexes

Let K be an abstract simplicial complex on the vertices $[r] = \{0, 1, \ldots, r\}$, i.e., K is a set of subsets of the set $[r]$ closed under inclusions;

$$G \subseteq F \in K \Rightarrow G \in K.$$

We can associate a simplicial set $K_\bullet$ to K as follows:

$$K_n = \{\psi \colon [n] \to [r] \in \Delta \mid \operatorname{im}(\psi) \subseteq K\}.$$

For $\varphi \colon [m] \to [n]$, we define $\varphi^* \colon K_n \to K_m$ by $\varphi^*(\psi) = \psi \circ \varphi$.

The *standard simplicial r-simplex* $\Delta[r]$ is the simplicial set associated to the simplicial complex $2^{[r]}$ consisting of all subsets of $[r]$. The boundary $\partial\Delta[r]$ is the simplicial set associated to the simplicial complex $2^{[r]} \setminus [r]$.

Proposition 6.2. 1. *Specifying a simplicial map $f \colon \Delta[n] \to X$ is equivalent to specifying a simplex $x \in X_n$.*

2. *Specifying a simplicial map $f \colon \partial\Delta[n] \to X$ is equivalent to specifying a collection of simplices*

$$f_{i_0 \dots i_k} \in X_k,$$

for all $0 \le i_0 < \cdots < i_k \le n$ where $k < n$, such that

$$d_j(f_{i_0 \dots i_k}) = f_{i_0 \dots \widehat{i_j} \dots i_k}$$

for all $0 \le j \le k$ and all $(i_0, \dots, i_k)$.

6.3 Geometric realization

Let X be a simplicial set. The geometric realization of X is the quotient space

$$|X| = \coprod_{n \ge 0} X_n \times \Delta^n / \sim,$$

where we make the identifications

$$(\varphi^*(x), \mathbf{t}) \sim (x, \varphi_*(\mathbf{t}))$$

for all morphisms $\varphi \colon [m] \to [n]$ in Δ, all n-simplices $x \in X_n$ and all $\mathbf{t} \in \Delta^m$.

The geometric realization is a functor $|-| \colon s\mathbf{Set} \to \mathbf{Top}$. It is left adjoint to the singular simplicial set functor $S_\bullet \colon \mathbf{Top} \to s\mathbf{Set}$, that is, there is a natural bijection

$$\mathbf{Top}(|X|, T) \cong s\mathbf{Set}(X, S_\bullet(T)), \tag{6}$$

for simplicial sets X and topological spaces T. In particular, geometric realization preserves all colimits.

By plugging in $X = S_\bullet(T)$ in (6), we obtain the unit of the adjunction, $\eta_T \colon |S_\bullet(T)| \to T$. A fundamental result in simplicial homotopy theory is the following.

Theorem 6.3. *For a topological space T, the unit map*

$$\eta_T \colon |S_\bullet(T)| \xrightarrow{\sim} T$$

is a natural weak homotopy equivalence.

If one is interested in studying topological spaces up to weak homotopy equivalence, then one might as well work with simplicial sets, because every topological space can be recovered up to weak homotopy equivalence from its singular simplicial set.

If K is an abstract simplicial complex, then the geometric realization of the simplicial set $|K_\bullet|$, as defined above, is naturally homeomorphic to the usual geometric realization of the simplicial complex K.

6.4 The skeletal filtration

Let X be a simplicial set. The *n-skeleton* of X is the subcomplex $X(n) \subseteq X$ generated by all non-degenerate simplices of X of dimension at most n. This gives a filtration

$$\emptyset = X(-1) \subseteq X(0) \subseteq X(1) \subseteq \cdots \subseteq \bigcup X(n) = X.$$

For every $n \geq 0$, there is a pushout diagram

$$\begin{tikzcd} \coprod_x \partial\Delta[n] \arrow[r] \arrow[d] & X(n-1) \arrow[d] \\ \coprod_x \Delta[n] \arrow[r] & X(n), \end{tikzcd} \tag{7}$$

where the disjoint unions are over all non-degenerate n-simplices x of X. By using the fact that geometric realizations commute with colimits, this can be used to prove that $|X|$ is a CW-complex.

We will say that X is n-dimensional if $X = X(n)$. In other words, X is n-dimensional if all simplices of dimension $> n$ are degenerate. If X is n-dimensional, then $|X|$ is an n-dimensional CW-complex. For example, the simplicial set $\Delta[n]$ is n-dimensional. The $(n-1)$-skeleton of $\Delta[n]$ is the simplicial set $\partial\Delta[n]$.

6.5 Simplicial abelian groups

Let A be a simplicial abelian group, i.e., a simplicial object in the category **Ab** of abelian groups. We can form a non-negatively graded chain complex

$$\cdots \to A_n \overset{\partial_n}{\to} A_{n-1} \to \cdots \to A_1 \overset{\partial_1}{\to} A_0$$

by letting

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

The simplicial identity $d_i d_j = d_{j-1} d_i$ for $i < j$ ensures that $\partial_{n-1}\partial_n = 0$. This chain complex will be denoted A_*.

Definition 6.4. The *normalized chain complex* associated to a simplicial abelian group A is the chain complex $N_*(A)$ with

$$N_n(A) = A_n / D_n(A)$$

where $D_n(A) \subseteq A_n$ is the subgroup generated by all elements of the form $s_i(a)$ for $a \in A_{n-1}$.

The projection $A_* \to N_*(A)$ induces an isomorphism in homology.

Theorem 6.5 (Dold-Kan correspondence)**.** *The normalized chain complex functor*

$$N_* \colon s\mathbf{Ab} \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$$

is an equivalence of categories.

Theorem 6.6. *Let A be a simplicial abelian group. For every point $* \in |A|$ and every $n \geq 0$, there is an isomorphism*

$$\pi_n(|A|, *) \cong \mathrm{H}_n(N_*(A)).$$

There is a functor $\mathbb{Z}\colon s\mathbf{Set} \to s\mathbf{Ab}$ which associates to a simplicial set X the simplicial abelian group $\mathbb{Z}X$ whose group of n-simplices,

$$(\mathbb{Z}X)_n = \mathbb{Z}X_n,$$

is the free abelian group on X_n. We shall write $C_*(X) = N_*\mathbb{Z}X$ for the *normalized chains* on X.

If T is a topological space, then $C_*(S_\bullet(T))$ is the singular chain complex of T.

If K is an abstract simplicial complex, then $C_*(K_\bullet)$ is isomorphic to the usual chain complex associated to K.

If X and K are simplicial sets, then let $X(K)$ denote the set of simplicial maps from K to X. If A is a simplicial abelian group, then the set $A(K)$ has an abelian group structure by defining addition pointwise.

Proposition 6.7. *The following are equivalent for a simplicial abelian group A.*

1. *The restriction map $A(K) \to A(L)$ is surjective for every inclusion $L \subset K$ of simplicial sets.*
2. *The restriction map $A(\Delta[n]) \to A(\partial\Delta[n])$ is surjective for every $n \geq 1$.*
3. *The normalized chain complex N_*A has zero homology.*

Proof. $1 \Rightarrow 2$ is trivial.

$2 \Rightarrow 3$: Let $a \in N_{n-1}A$ be a cycle in the normalized chain complex, i.e., an $(n-1)$-simplex $a \in A_{n-1}$ such that $d_i a = 0$ for all i. We can specify a map $f\colon \partial\Delta[n] \to A$ by setting $f_{1\ldots n} = a$ and $f_{i_0\ldots i_k} = 0$ for all other sequences $0 \leq i_0 < \ldots < i_k \leq n$ with $k < n$. By 2 this map extends to $\widehat{f}\colon \Delta[n] \to A$. Let $b = \widehat{f}_{0\ldots n}$. Then $d_i(b) = 0$ for all $i > 0$, so that $b \in N_n A$, and moreover $d_0 b = a$, which shows that a is a boundary in N_*A.

$3 \Rightarrow 1$: If A is a simplicial abelian group, then there is a natural bijection

$$s\mathbf{Set}(X, A) \cong \mathbf{Ch}_{\geq 0}(\mathbb{Z})(C_*(X), N_*A)$$

by the Dold-Kan correspondence. Thus, we need to show that for every inclusion of simplicial sets $L \subset K$, every morphism of chain complexes

$$f\colon C_*(L) \to N_*A$$

extends to $C_*(K)$. Now, morphisms of chain complexes $C_* \to D_*$ are precisely zero-cycles of the chain complex $\mathrm{Hom}(C_*, D_*)$, so we need to show that the restriction map

$$Z_0 \operatorname{Hom}(C_*(K), N_*A) \to Z_0 \operatorname{Hom}(C_*(L), N_*A) \tag{8}$$

is surjective. The quotient $C_*(K)/C_*(L)$ is a non-negatively graded chain complex of free modules. Therefore, we get a short exact sequence

$$0 \to \mathrm{Hom}(C_*(K)/C_*(L), N_*A) \to \mathrm{Hom}(C_*(K), N_*A) \to \mathrm{Hom}(C_*(L), N_*A) \to 0.$$

That $C_*(K)/C_*(L)$ is free and bounded below also implies that the functor $\mathrm{Hom}(C_*(K)/C_*(L), -)$ preserves quasi-isomorphisms. In particular, if N_*A has zero homology, then so has $\mathrm{Hom}(C_*(K)/C_*(L), N_*A)$. It follows that the surjective morphism of chain complexes

$$\mathrm{Hom}(C_*(K), N_*A) \to \mathrm{Hom}(C_*(L), N_*A)$$

is a quasi-isomorphism. The surjectivity of (8) then follows from the lemma below. □

Lemma 6.8. *Let $C_* \to D_*$ be a surjective quasi-isomorphism of chain complexes. Then $Z_0C_* \to Z_0D_*$ is surjective.*

Proof. A zero-cycle in a chain complex C_* is the same as a morphism of chain complexes $\mathbb{Z} \to C_*$, where $\mathbb{Z}$ is viewed as a chain complex concentrated in degree 0. Then surjectivity of $Z_0C_* \to Z_0D_*$ follows from the lifting property of bounded below complexes of projective modules.

$$\begin{tikzcd} & C_* \arrow[d] \\ \mathbb{Z} \arrow[r] \arrow[ur, dashed] & D_* \end{tikzcd}$$

□

6.6 The Eilenberg-Zilber theorem

Let A and B be simplicial abelian groups. The tensor product $A \otimes B$ is defined to be the simplicial abelian group with n-simplices

$$(A \otimes B)_n = A_n \otimes B_n.$$

We can take its associated normalized chain complex $N_*(A \otimes B)$. On the other hand we can first form the normalized chain complexes of A and B , and then take the tensor product of chain complexes $N_*(A) \otimes N_*(B)$. Recall that

$$(N_*(A) \otimes N_*(B))_n = \bigoplus_{p+q=n} N_p(A) \otimes N_q(B).$$

In general, $N_*(A \otimes B)$ and $N_*(A) \otimes N_*(B)$ are not isomorphic. However, they are naturally chain homotopy equivalent.

Theorem 6.9 (Eilenberg-Zilber theorem)**.** *There is a natural chain homotopy equivalence*

$$AW\colon N_*(A \otimes B) \xrightarrow{\simeq} N_*(A) \otimes N_*(B).$$

The map AW, called the Alexander-Whitney map*, is defined by*

$$AW(a \otimes b) = \sum_{i=0}^{n} a_{0\dots i} \otimes b_{i\dots n}$$

for $a \in A_n$ and $b \in B_n$.

Corollary 6.10. *Let T and U be topological spaces. There is a natural chain homotopy equivalence*

$$C_*(T \times U) \xrightarrow{\simeq} C_*(T) \otimes C_*(U).$$

Proof. We apply the Eilenberg-Zilber theorem to the simplicial abelian groups $A = \mathbb{Z}S_\bullet(T)$ and $B = \mathbb{Z}S_\bullet(U)$, noting that $A \otimes B \cong \mathbb{Z}S_\bullet(T \times U)$. $\square$

7 Cochain algebras

This section is based on [3] and [5, II.10].

Let $\Bbbk$ be a commutative ring. In what follows, everything will be a module over $\Bbbk$ and tensor products will be taken over $\Bbbk$ etc.

Definition 7.1. A *cochain algebra* A^* consists of

- A *differential*
$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \to \cdots,$$
- A *product*
$$A^p \otimes A^q \to A^{p+q}, \quad x \otimes y \mapsto xy,$$
- A *unit* $1 \in A^0$,

subject to the following axioms:

1. (Associativity) $(xy)z = x(yz)$.
2. (Unitality) $1x = x1 = x$.
3. (Differential) $d^2 = 0$.
4. (Leibniz rule) $d(xy) = d(x)y + (-1)^p x d(y)$.

A cochain algebra A^* is called *(graded) commutative* if

$$xy = (-1)^{pq} yx$$

for all $x \in A^p$ and $y \in A^q$.

The Leibniz rule ensures that the *cohomology* $\mathrm{H}^*(A^*) = \ker d / \operatorname{im} d$ inherits a graded algebra structure from A^*.

A morphism of cochain algebras $g\colon A^* \to B^*$ is a collection of $\Bbbk$-linear maps $g^n\colon A^n \to B^n$ that commute with differentials and products, and that preserve the unit.

7.1 The normalized cochain algebra of a simplicial set

Let X be a simplicial set and let $\Bbbk$ be a commutative ring. The normalized cochain algebra $C^*(X;\Bbbk)$ is defined by

$$C^n(X;\Bbbk) = \{f\colon X_n \to \Bbbk \mid f(\textit{degenerate}) = 0\},$$

the $\Bbbk$-module of all functions from X_n to $\Bbbk$ that vanish on degenerate simplices.

- The differential $d\colon C^n(X;\Bbbk) \to C^{n+1}(X;\Bbbk)$ is defined by the formula

$$d(f)(x) = \sum_{i=0}^{n+1} (-1)^{i+n} f(d_i(x)),$$

  for $x \in X_{n+1}$.

- The product, known as the *cup product*,

$$\smile\colon C^p(X;\Bbbk) \otimes C^q(X;\Bbbk) \to C^{p+q}(X;\Bbbk),$$

  is defined by the formula

$$(f \smile g)(x) = f(x_{0\ldots p})g(x_{p\ldots p+q}),$$

  for $x \in X_{p+q}$.

- The unit $1 \in C^0(X;\Bbbk)$ is the constant function $X_0 \to \Bbbk$ with value the unit of $\Bbbk$.

The reader should check that these definitions make $C^*(X;\Bbbk)$ into a cochain algebra. In general, the cochain algebra $C^*(X;\Bbbk)$ is not graded commutative, but it is commutative up to homotopy. There is a binary operation

$$\smile_1\colon C^p(X;\Bbbk) \otimes C^q(X;\Bbbk) \to C^{p+q-1}(X;\Bbbk),$$

known as the *cup-1-product* which has the property that

$$f \smile g - (-1)^{pq} g \smile f = d(f \smile_1 g) + d(f) \smile_1 g + (-1)^p f \smile_1 d(g), \quad (9)$$

for all $f \in C^p(X;\Bbbk)$ and all $g \in C^q(X;\Bbbk)$. The cup-1-product is a contracting homotopy for the commutator of the cup product; it exhibits the cup product as a *homotopy commutative* operation. If f and g are cocycles, then (9) shows that $f \smile g$ is cohomologous to $(-1)^{pq} g \smile f$. In particular, this proves that the induced product in the cohomology $\mathrm{H}^*(X;\Bbbk)$ is graded commutative. Here is a formula for the cup-1-product:

$$f \smile_1 g = \sum_{i<j} \pm f(x_{0\ldots ij\ldots p+q-1}) g(x_{i\ldots j}).$$

There is also a *cup-i-product* for every $i \geq 1$. It is a binary operation

$$\cup_i\colon C^p(X;\Bbbk) \otimes C^q(X;\Bbbk) \to C^{p+q-i}(X;\Bbbk).$$

The cup-i-products were introduced by Steenrod [24], and he used them to define what are nowadays called the Steenrod operations. For $\Bbbk = \mathbb{F}_2$, the Steenrod operation

$$\mathrm{Sq}^i\colon \mathrm{H}^p(X;\mathbb{F}_2) \to \mathrm{H}^{p+i}(X;\mathbb{F}_2)$$

is defined by

$$\mathrm{Sq}^i[f] = [f \cup_{p-i} f]$$

for p-cocycles f and $0 \leq i \leq p$.

Remark 7.2. There are multivariable generalizations of the cup-i-products that endow $C^*(X;\Bbbk)$ with the structure of an *E_∞-algebra*, or a *strongly homotopy commutative algebra*. This is the topic of a beautiful paper by McClure and Smith [19].

The problem of commutative cochains

The cohomology algebra of a simplicial set is graded commutative. It is therefore natural to ask whether it is possible to find a graded commutative cochain algebra $\mathscr{A}^*(X)$ that depends functorially on X with the property that $\mathscr{A}^*(X)$ is weakly equivalent to $C^*(X;\Bbbk)$ as a cochain algebra.

This is not possible in general. For instance, if $\Bbbk = \mathbb{F}_2$, the non-triviality of the Steenrod operations is an obstruction to the existence of commutative cochains. However, if $\Bbbk$ is a field of characteristic zero, there is no obstruction, and it turns out that it is possible to find commutative cochains in this case. In what follows, we will give an axiomatic characterization of when a cochain algebra functor $\mathscr{A}^*(X)$ is equivalent to $C^*(X;\Bbbk)$. Then we will describe Sullivan's commutative cochain algebra of polynomial differential forms, which is a solution to the commutative cochain algebra problem when $\Bbbk$ contains $\mathbb{Q}$ as a subring.

7.2 Cochain algebra functors from simplicial cochains algebras

Let $\mathscr{A} = \mathscr{A}_\bullet^*$ be a simplicial cochain algebra, i.e., a functor

$$\mathscr{A}\colon \Delta^{op} \to \mathbf{DGA}.$$

There is a canonically associated contravariant functor

$$\mathscr{A}^*(-)\colon s\mathbf{Set} \to \mathbf{DGA}$$

determined by the two properties

- $\mathscr{A}^*(\Delta[n]) = \mathscr{A}_n^*$.
- $\mathscr{A}(-)$ takes colimits to limits.

We define $\mathscr{A}^p(X)$ to be the set of simplicial maps from X to the simplicial abelian group $\mathscr{A}_\bullet^p$. Concretely, a p-cochain $f \in \mathscr{A}^p(X)$ consists of a collection of p-cochains

$$f_x \in \mathscr{A}_n^p,$$

one for every n-simplex $x \in X_n$, that are compatible in the sense that

$$\varphi^*(f_x) = f_{\varphi^*(x)}$$

for every morphism $\varphi\colon [m] \to [n]$ in Δ. The differential, product and unit are defined pointwise;

$$\begin{aligned} d(f)_x &= d(f_x), \\ (fg)_x &= f_x g_x, \\ 1_x &= 1. \end{aligned}$$

The construction $\mathscr{A}^*(X)$ is functorial in both X and $\mathscr{A}$. If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a morphism of simplicial cochain algebras, then for every simplicial set X there is an induced morphism of cochain algebras

$$\Phi(X)\colon \mathscr{A}^*(X) \to \mathscr{B}^*(X)$$

sending a cochain $f = \{f_x\}_{\in X}$ to the cochain $\Phi(X)(f) = \{\Phi(X)(f)_x = \Phi(f_x)\}_{x\in X}$. Given a morphism of simplicial sets $g\colon X \to Y$, there is an induced morphism of cochain algebras

$$g^*\colon \mathscr{A}^*(Y) \to \mathscr{A}^*(X)$$

given by

$$g^*(f)_x = f_{g(x)}.$$

Example: The normalized cochain algebra

The normalized cochain algebra, discussed in the previous section, arises from a simplicial cochain algebra in the following way. Consider the simplicial cochain algebra $\mathscr{C}^*_\bullet$ with n-simplices

$$\mathscr{C}^*_n = C^*(\Delta[n]; \Bbbk),$$

the normalized cochain algebra on $\Delta[n]$. Since $C^*(-;\Bbbk)$ takes colimits to limits, it follows that

$$\mathscr{C}^*_\bullet(X) \cong C^*(X;\Bbbk)$$

for every simplicial set X.

7.3 Extendable cochain algebras

Theorem 7.3. *The following are equivalent for a simplicial cochain algebra $\mathscr{A}$.*

1. *The restriction map $\mathscr{A}^*(K) \to \mathscr{A}^*(L)$ is surjective for every inclusion of simplicial sets $L \subset K$.*
2. *The restriction map $\mathscr{A}^*(\Delta[n]) \to \mathscr{A}^*(\partial\Delta[n])$ is surjective for every $n \geq 0$.*
3. *The homology groups of the normalized chain complex $N_*(\mathscr{A}^p_\bullet)$ are all zero, for every p.*

Proof. This is a direct consequence of Propostion 6.7. □

Definition 7.4. A simplicial cochain algebra $\mathscr{A}$ is called *extendable* if the equivalent conditions in Theorem 7.3 are satisfied.

As an example, the simplicial cochain algebra $\mathscr{C}^*_\bullet$, whose associated cochain algebra is the normalized cochain algebra, is extendable. Indeed, given an inclusion of simplicial sets $L \subset K$, the restriction map

$$C^p(K;\Bbbk) \to C^p(L;\Bbbk)$$

is surjective for every p, because a normalized p-cochain $f\colon L_p \to \Bbbk$ can be extended to $\widetilde{f}\colon K_p \to \Bbbk$, e.g., by setting

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in L_p, \\ 0, & x \notin L_p, \end{cases}$$

for $x \in K_p$.

If $\mathscr{A}$ is a simplicial cochain algebra, then for an inclusion $L \subset K$ of simplicial sets, we define $\mathscr{A}(K, L)$ to be the kernel of the restriction map. Thus, if $\mathscr{A}$ is extendable, then there is a short exact sequence of cochain complexes

$$0 \to \mathscr{A}^*(K, L) \to \mathscr{A}^*(K) \to \mathscr{A}^*(L) \to 0.$$

This should be familiar; for $\mathscr{A}^*(-)$ equal to the normalized cochain algebra, this is what one uses to derive the basic properties of relative cohomology.

Definition 7.5. A simplicial cochain algebra $\mathscr{A}$ is said to *satisfy the Poincaré lemma* if the unit map $\Bbbk \to \mathscr{A}^*(\Delta[n])$ is a quasi-isomorphism for every $n \geq 0$. In other words, $\mathscr{A}$ satisfies the Poincaré lemma if and only if

$$\mathrm{H}^p(\mathscr{A}^*(\Delta[n])) = \begin{cases} \Bbbk, & p = 0 \\ 0, & p > 0. \end{cases}$$

Clearly, the normalized cochain algebra $C^*(-; \Bbbk)$ satisfies the Poincaré lemma.

Lemma 7.6. *Let $\mathscr{A}^*$ be a simplicial cochain algebra and X a simplicial set. There is a natural isomorphism*

$$\mathscr{A}^*(X(n), X(n-1)) \cong \prod_x \mathscr{A}^*(\Delta[n], \partial\Delta[n]),$$

where the product is over all non-degenerate n-simplices x of X.

Proof. The functor $\mathscr{A}^*(-)$ takes colimits to limits. When we apply it to the diagram (7), we therefore obtain a pullback diagram

$$\begin{tikzcd} \mathscr{A}^*(X(n)) \arrow[r] \arrow[d] & \prod_x \mathscr{A}^*(\Delta[n]) \arrow[d] \\ \mathscr{A}^*(X(n-1)) \arrow[r] & \prod_x \mathscr{A}^*(\partial\Delta[n]). \end{tikzcd}$$

It follows that the kernels of the two vertical maps are isomorphic. □

Proposition 7.7. *The following are equivalent for a morphism $\Phi\colon \mathscr{A} \to \mathscr{B}$ of extendable simplicial cochain algebras.*

1. *The map $\Phi_n\colon \mathscr{A}_n \to \mathscr{B}_n$ is a quasi-isomorphism for every $n \geq 0$.*
2. *The map $\Phi(K)\colon \mathscr{A}(K) \to \mathscr{B}(K)$ is a quasi-isomorphism for every simplicial set K.*

Proof. $\Leftarrow$: Plug in $K = \Delta[n]$.

$\Rightarrow$: <u>For finite dimensional K</u>: We will first prove that $\Phi(K)$ is a quasi-isomorphism for all finite dimensional K by induction on the dimension. For K of dimension -1, i.e., $K = \emptyset$, the statement is vacuous. Let $n \geq 0$, and assume by induction that $\Phi(L)$ is a quasi-isomorphism for all simplicial sets L of dimension $< n$. Let K be a simplicial set of dimension n, i.e., $K = K(n)$. Since both $\mathscr{A}$ and $\mathscr{B}$ are extendable, we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{A}^*(K(n), K(n-1)) \arrow[r] \arrow[d] & \mathscr{A}^*(K(n)) \arrow[r] \arrow[d] & \mathscr{A}^*(K(n-1)) \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & \mathscr{B}^*(K(n), K(n-1)) \arrow[r] & \mathscr{B}^*(K(n)) \arrow[r] & \mathscr{B}^*(K(n-1)) \arrow[r] & 0 \end{tikzcd} \tag{10}$$

The right vertical map in (10) is a quasi-isomorphism by the induction hypothesis. We want to prove that the middle map is a quasi-isomorphism. To do this, it suffices to prove that the left map is a quasi-isomorphism, by the five lemma. By Lemma 7.6, the left map is isomorphic to the product map

$$\prod_x \mathscr{A}^*(\Delta[n], \partial\Delta[n]) \to \prod_x \mathscr{B}^*(\Delta[n], \partial\Delta[n]). \tag{11}$$

Each component of this map appears to the left in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{A}^*(\Delta[n], \partial\Delta[n]) \arrow[r] \arrow[d] & \mathscr{A}^*(\Delta[n]) \arrow[r] \arrow[d, "\sim"] & \mathscr{A}^*(\partial\Delta[n])) \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \mathscr{B}^*(\Delta[n], \partial\Delta[n]) \arrow[r] & \mathscr{B}^*(\Delta[n]) \arrow[r] & \mathscr{B}^*(\partial\Delta[n]) \arrow[r] & 0
\end{tikzcd}$$

In this diagram, the right map is a quasi-isomorphism by induction, and the middle map is a quasi-isomorphism by hypothesis. It follows that the left vertical map is a quasi-isomorphism. Hence (11) is a quasi-isomorphism as well. This finishes the induction.

For arbitrary K: The skeletal filtration of K gives a morphism of towers of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{A}^*(K(n)) \arrow[r] \arrow[d, "\sim"] & \mathscr{A}^*(K(n-1)) \arrow[r] \arrow[d, "\sim"] & \cdots \arrow[r] & \mathscr{A}^*(K(0)) \arrow[d, "\sim"] \\
\cdots \arrow[r] & \mathscr{B}^*(K(n)) \arrow[r] & \mathscr{B}^*(K(n-1)) \arrow[r] & \cdots \arrow[r] & \mathscr{B}^*(K(0))
\end{tikzcd}$$

where the vertical maps are quasi-isomorphisms by the first part of the proof. Since $K = \cup_n K(n)$ and $\mathscr{A}^*(-)$ takes colimits to limits, we have a natural isomorphism $\mathscr{A}^*(K) \cong \varprojlim \mathscr{A}^*(K(n))$. The claim follows from Lemma 7.8 below. $\square$

Lemma 7.8. *Let $\{f(n)\}\colon \{A(n)\}_{n\geq 0} \to \{B(n)\}_{n\geq 0}$ be a morphism of towers of chain complexes. If each component $f(n)\colon A(n) \to B(n)$ is a quasi-isomorphism, then the induced map $\varprojlim A(n) \to \varprojlim B(n)$ is a quasi-isomorphism.*

Tensor products of simplicial cochain algebras

The tensor product $\mathscr{A} \otimes \mathscr{B}$ of two simplicial cochain algebras $\mathscr{A}$ and $\mathscr{B}$ is the simplicial cochain algebra with

$$(\mathscr{A} \otimes \mathscr{B})_n^p = \bigoplus_{s+t=p} \mathscr{A}_n^s \otimes \mathscr{B}_n^t$$

and the obvious structure.

Proposition 7.9. *Let $\mathscr{A}$ and $\mathscr{B}$ be simplicial cochain algebras.*

1. *If $\mathscr{A}$ and $\mathscr{B}$ are extendable, then so is $\mathscr{A} \otimes \mathscr{B}$.*
2. *If $\mathscr{A}$ and $\mathscr{B}$ satisfy the Poincaré lemma, then so does $\mathscr{A} \otimes \mathscr{B}$.*

Proof. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are extendable. To prove that $\mathscr{A} \otimes \mathscr{B}$ is extendable, we will use the third characterization from Theorem 7.3. The normalized cochain functor N_* commutes with direct sums, so for every p we have

$$N_*((\mathscr{A} \otimes \mathscr{B})^p_\bullet) \cong \bigoplus_{s+t=p} N_*(\mathscr{A}^s_\bullet \otimes \mathscr{B}^t_\bullet).$$

By the Eilenberg-Zilber theorem, there is a chain homotopy equivalence

$$N_*(\mathscr{A}^s_\bullet \otimes \mathscr{B}^t_\bullet) \xrightarrow{\simeq} N_*(\mathscr{A}^s_\bullet) \otimes N_*(\mathscr{B}^t_\bullet).$$

Since $\mathscr{A}$ and $\mathscr{B}$ are extendable, the chain complex to the right has zero homology. This proves the first claim.

Next, suppose that $\mathscr{A}$ and $\mathscr{B}$ satisfy the Poincaré lemma. Then by the Künneth theorem

$$\begin{aligned} \mathrm{H}^*((\mathscr{A} \otimes \mathscr{B})(\Delta[n])) &= \mathrm{H}^*(\mathscr{A}_n \otimes \mathscr{B}_n) \\ &\cong \mathrm{H}^*(\mathscr{A}_n) \otimes \mathrm{H}^*(\mathscr{B}_n). \end{aligned}$$

This cochain complex has the correct cohomology. □

Theorem 7.10. *If $\mathscr{A}$ and $\mathscr{B}$ are extendable simplicial cochain algebras that satisfy the Poincaré lemma, then there are natural quasi-isomorphisms of cochain algebras*

$$\mathscr{A}^*(X) \xrightarrow{\sim} (\mathscr{A} \otimes \mathscr{B})^*(X) \xleftarrow{\sim} \mathscr{B}^*(X)$$

for every simplicial set X. In particular, there is a natural isomorphism of graded algebras

$$\mathrm{H}^*(\mathscr{A}^*(X)) \cong \mathrm{H}^*(\mathscr{B}^*(X)).$$

Proof. There are natural morphisms of simplicial cochain algebras

$$\mathscr{A}^*_\bullet \xrightarrow{\sim} (\mathscr{A} \otimes \mathscr{B})^*_\bullet \xleftarrow{\sim} \mathscr{B}^*_\bullet.$$

By Proposition 7.9, all simplicial cochain algebras above are extendable, so by Proposition 7.7 it suffices to prove the claim for $X = \Delta[n]$. This means that we should prove that we have quasi-isomorphisms

$$\mathscr{A}^*_n \xrightarrow{\sim} \mathscr{A}^*_n \otimes \mathscr{B}^*_n \xleftarrow{\sim} \mathscr{B}^*_n$$

for every n. But this follows immediately since we know that $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{A} \otimes \mathscr{B}$ satisfy the Poincaré lemma. □

7.4 The simplicial de Rham algebra

The simplicial de Rham algebra is the *graded commutative* simplicial cochain algebra $\Omega^*_\bullet$ with n-simplices

$$\Omega^*_n = \frac{\Bbbk[t_0, \ldots, t_n] \otimes \Lambda(dt_0, \ldots, dt_n)}{(t_0 + \cdots + t_n - 1, dt_0 + \cdots + dt_n)}, \quad |t_i| = 0, \quad |dt_i| = 1.$$

This should be thought of as the cochain algebra of polynomial differential forms on an n-simplex. The differential $d\colon \Omega_n^* \to \Omega_n^{*+1}$ is determined by the formula

$$d(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial t_i} dt_i,$$

for $f \in \Bbbk[t_0, \ldots, t_n]/(\sum_i t_i - 1)$, and the Leibniz rule. The simplicial structure is described as follows: given a morphism $\varphi\colon [m] \to [n]$ in Δ, the morphism of cochain algebras $\varphi^*\colon \Omega_n^* \to \Omega_m^*$ is determined by

$$\varphi^*(t_i) = \sum_{j \in \varphi^{-1}(i)} t_j. \tag{12}$$

Exercise 7.11. Check that (12) gives a well-defined morphism of cochain algebras.

Proposition 7.12. *The simplicial de Rham algebra $\Omega_\bullet^*$ satisfies the Poincaré lemma if and only if the ground ring $\Bbbk$ is uniquely divisible as an abelian group (i.e., $\mathbb{Q} \subseteq \Bbbk$).*

Proof. There is an isomorphism of cochain algebras

$$\Omega_n^* \cong \Bbbk[t_1, \ldots, t_n] \otimes \Lambda(dt_1, \ldots, dt_n) \cong (\Bbbk[t] \otimes \Lambda(dt))^{\otimes n},$$

for every $n \geq 0$. By the Künneth theorem we have an isomorphism

$$\mathrm{H}^*(\Omega_n^*) \cong \mathrm{H}^*(\Bbbk[t] \otimes \Lambda(dt))^{\otimes n}.$$

Here is a picture of the cochain complex $\Bbbk[t] \otimes \Lambda(dt)$:

$$\begin{tikzcd}
1 & t \arrow[d, "\cdot 1"] & t^2 \arrow[d, "\cdot 2"] & t^3 \arrow[d, "\cdot 3"] & t^4 \arrow[d, "\cdot 4"] & \cdots \\
 & dt & tdt & t^2dt & t^3dt & \cdots
\end{tikzcd}$$

This picture shows that the cochain algebra $\Bbbk[t] \otimes \Lambda(dt)$ has cohomology $\Bbbk$ concentrated in degree 0 if and only if $\Bbbk \xrightarrow{\cdot m} \Bbbk$ is an isomorphism for all $m > 0$. $\square$

Proposition 7.13. *The simplicial de Rham algebra $\Omega_\bullet^*$ is extendable.*

Proof. By Theorem 7.3, it suffices to check that the normalized chain complex $N_*(\Omega_\bullet^p)$ has zero homology for every p. Let $\omega \in N_n(\Omega_\bullet^p)$ be an n-cycle. This means that $\omega \in \Omega_n^p$ is a p-cochain such that $\partial_i(\omega) = 0$ for all $0 \leq i \leq n$. We need to show that ω is a boundary, i.e., that there exists $\nu \in \Omega_{n+1}^p$ such that $\partial_i(\nu) = 0$ for $0 < i \leq n$ and $\partial_0(\nu) = \omega$. We claim that

$$\nu := \sum_{j=1}^{n+1} t_j \omega(t_1, \ldots, t_{j-1}, t_j + t_0, t_{j+1}, \ldots, t_{n+1})$$

does the trick. The verification is left to the reader. $\square$

Theorem 7.14. *If the ground ring $\Bbbk$ is uniquely divisible (i.e., $\mathbb{Q} \subseteq \Bbbk$), then there are natural quasi-isomorphisms of cochain algebras*

$$\Omega^*(X) \xrightarrow{\sim} (\Omega \otimes \mathscr{C})^*(X) \xleftarrow{\sim} C^*(X; \Bbbk)$$

for all simplicial sets X.

Proof. Under the stated hypothesis, both simplicial cochain algebras $\Omega^*_\bullet$ and $\mathscr{C}^*_\bullet$ are extendable and satisfy the Poincaré lemma, so the claim follows from Theorem 7.10. □

This solves the problem of finding commutative cochains over rings $\Bbbk$ that contain $\mathbb{Q}$ as a subring.

8 Homotopy theory of commutative cochain algebras

This section is based on material from [1] and [5].

8.1 Relative Sullivan algebras

Relative Sullivan algebras are the cochain algebra analogs of relative CW-complexes. To define them, we first need to discuss the cochain analogs of spheres and disks.

Spheres and disks

Let n be a positive integer.

Let $S(n)$ denote the cochain algebra

$$S(n) = (\Lambda x, dx = 0), \quad |x| = n.$$

In other words, the underlying algebra of $S(n)$ is the free graded commutative algebra on a generator x of degree n, and the differential is zero.

Let $D(n-1)$ denote the cochain algebra

$$D(n-1) = (\Lambda(x, sx), dx = 0, d(sx) = x), \quad |x| = n, \ |sx| = n-1.$$

Note that there is an obvious inclusion $S(n) \subset D(n-1)$ of cochain algebras.

Exercise 8.1. Prove that there are natural bijections

$$\mathrm{Hom}_{dga}(S(n), A) \cong Z^n(A),$$

$$\mathrm{Hom}_{dga}(D(n-1), A) \cong A^{n-1},$$

for cochain algebras A.

It will be convenient to have a slightly more general construction. For a graded vector space V, define cochain algebras

$$S(V) = (\Lambda(V), dv = 0), \quad v \in V$$

$$D(V) = (\Lambda(V \oplus sV), d(v) = 0, d(sv) = v), \quad v \in V$$

There is an inclusion of cochain algebras $S(V) \subset D(V)$. If V is one-dimensional and concentrated in degree n, then $S(V) = S(n)$ and $D(V) = D(n-1)$. More generally, if $\{x_i\}_{i \in I}$ is a homogeneous basis for V, then

$$S(V) \cong \bigotimes_{i \in I} S(|x_i|), \quad D(V) \cong \bigotimes_{i \in I} D(|x_i|).$$

Exercise 8.2.

1. Prove that there are natural bijections
$$\mathrm{Hom}_{dga}(S(V), A) \cong \mathrm{Hom}_{\Bbbk}(V, Z^*(A)),$$
$$\mathrm{Hom}_{dga}(D(V), A) \cong \mathrm{Hom}_{\Bbbk}(sV, A),$$
for graded vectors spaces V and cochain algebras A.
2. Prove that the unit map $\eta\colon \Bbbk \to D(V)$ is a quasi-isomorphism for every graded vector space V if and only if $\mathbb{Q} \subseteq \Bbbk$.

Attaching generators to kill cocycles

Let A be a cochain algebra and let $a \in A^n$ be an n-cocycle. We define a new cochain algebra

$$A[y|dy = a]$$

as follows. The underlying algebra is the tensor product $A \otimes \Lambda(y)$, where y is a generator of degree $n-1$. The differential is determined by the formulas

$$d(b \otimes 1) = d(b) \otimes 1, \quad d(1 \otimes y) = a \otimes 1$$

and the Leibniz rule. We say that $A[y|dy = a]$ is obtained from A by 'adding a generator y to kill the cocycle a'. Equivalently, $A[y|dy = a]$ is obtained as a pushout

$$\begin{tikzcd}
S(n) \arrow[r, "a"] \arrow[d] & A \arrow[d] \\
D(n-1) \arrow[r, "y"] & A[y|dy = a]
\end{tikzcd}$$

Definition 8.3. A *relative Sullivan algebra* is a pair of cochain algebras (X, A) together with a filtration

$$A = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$$

such that $X = \cup_n X_n$ and each X_n is obtained from X_{n-1} by 'attaching cells'. This means that for each $n \geq 0$, there is a graded vector space V_n and a pushout diagram in the category of cochain algebras

$$\begin{tikzcd}
S(V_n) \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
D(V_n) \arrow[r] & X_n.
\end{tikzcd}$$

A *Sullivan algebra* is a cochain algebra X such that the pair $(X, \Bbbk)$ (where $\Bbbk$ includes into X by the unit map) admits the structure of a relative Sullivan algebra.

Example 8.4. • For every graded vector space V, the 'sphere' $S(V)$ is a Sullivan algebra: there is a pushout diagram

$$\begin{tikzcd} S(s^{-1}V) \arrow[r] \arrow[d] & \Bbbk \arrow[d] \\ D(s^{-1}V) \arrow[r] & S(V), \end{tikzcd}$$

so we can take $X_{-1} = \Bbbk \subset X_0 = S(V)$ and $V_0 = s^{-1}V$.

- The 'disk' $D(V)$ is a Sullivan algebra. We can take the filtration $\Bbbk \subset S(V) \subset D(V)$, and $V_0 = s^{-1}V$, $V_1 = V$.
- Let A be a cochain algebra and $a \in A^n$ a cocycle. Then $(A[y|dy = a], A)$ is a relative Sullivan algebra.

If (X, A) is a relative Sullivan algebra, then it follows that there is an isomorphism of graded algebras $X \cong A \otimes \Lambda(sV)$, where $V = \oplus_n V_n$. In particular, if X is a Sullivan algebra, then the underlying algebra of X is a free graded commutative algebra. The converse is false; the following exercise shows that not every cochain algebra whose underlying algebra is free is a Sullivan algebra.

Exercise 8.5. Consider the cochain algebra

$$X = (\Lambda(x, y, z),\ dx = yz,\ dy = xz,\ dz = xy), \quad |x| = |y| = |z| = 1.$$

Prove that X is not a Sullivan algebra.

Definition 8.6. Let $f\colon A \to B$ be a morphism of cochain algebras. A *Sullivan model for f* is a relative Sullivan algebra (X, A) together with a quasi-isomorphism $\widetilde{f}\colon X \xrightarrow{\sim} B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & X \arrow[ur, "\widetilde{f}"', "\sim"] & \end{tikzcd}$$

A *Sullivan model* for a cochain algebra B is a Sullivan model for the unit map $\eta\colon \Bbbk \to B$, i.e., a Sullivan algebra X together with a quasi-isomorphism $\widetilde{\eta}\colon X \xrightarrow{\sim} B$.

The following theorem is a cochain algebra version of the relative CW approximation theorem (cf. [10, Proposition 4.13]).

Theorem 8.7. *Let $f\colon A \to B$ be a morphism of cochain algebras such that $\mathrm{H}^0(A) \cong \mathrm{H}^0(B) \cong \Bbbk$ and $\mathrm{H}^1(f)\colon \mathrm{H}^1(A) \to \mathrm{H}^1(B)$ is injective. Then there is a Sullivan model (X, A) for f.*

Proof. □

Corollary 8.8. *Every cochain algebra B with $\mathrm{H}^0(B) \cong \Bbbk$ admits a Sullivan model X.*

Exercise 8.9. Let x and z be generators of degree 2 and 4, respectively, and consider the following morphism of cochain algebras:

$$f\colon \mathbb{Q}[z] \to \mathbb{Q}[x], \quad z \mapsto x^2.$$

1. Construct explicitly a Sullivan model of f,

$$\begin{tikzcd} \mathbb{Q}[z] \arrow[rr, "f"] \arrow[dr, tail] & & \mathbb{Q}[x] \\ & X \arrow[ur, "\sim", "\widetilde{f}"'] & \end{tikzcd}$$

2. Calculate the cohomology algebra $\mathrm{H}^*(C)$, where C is the 'homotopy cofiber', $C = \mathbb{Q} \otimes_{\mathbb{Q}[z]} X$.

Cofibrations

Definition 8.10. A morphism of cochain algebras $i\colon A \to X$ is called a *cofibration* if in every commutative square of cochain algebras

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, two heads, "\sim", "\pi"'] \\ X \arrow[r] \arrow[ur, dashed, "\lambda"] & C \end{tikzcd}$$

with π a surjective quasi-isomorphism, there exists a morphism λ making the diagram commute.

A cochain algebra X is called *cofibrant* if the unit map $\eta\colon \Bbbk \to X$ is a cofibration.

Theorem 8.11. *If (X, A) is a relative Sullivan algebra, then the inclusion morphism $A \to X$ is a cofibration. In particular, every Sullivan algebra X is cofibrant.*

The proof of this theorem will be done in two steps. The first step is completely formal and uses no particular properties of cochain algebras.

Proposition 8.12. *1. Given a family of cofibrations $i_j\colon A_j \to X_j$ the coproduct $\otimes_j A_j \to \otimes_j X_j$ is a cofibration.*

2. Given a pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & B \arrow[d, "j"] \\ X \arrow[r] & Y \end{tikzcd}$$

if i is a cofibration, then so is j.

3. Given a sequence of maps

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \cdots$$

if every i_j is a cofibration, then so is the induced map $X_0 \to \operatorname{colim}_n X_n$.

Proof. We will prove the second statement and leave the other two to the reader. Let there be given a surjective quasi-isomorphism $p\colon D \to E$ and morphisms b and y such that the right square below is commutative. Our task is then to find a morphism μ as indicated such that $p \circ \mu = y$ and $\mu \circ j = b$.

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] & B \arrow[r, "b"] \arrow[d, "j"] & D \arrow[d, two heads, "p", "\sim"'] \\
X \arrow[r, "x"'] \arrow[urr, dashed, "\lambda"] & Y \arrow[r, "y"'] \arrow[ur, dashed, "\mu"'] & E
\end{tikzcd}
$$

Since i is a cofibration, we can find a morphism λ such that $p \circ \lambda = y \circ x$ and $\lambda \circ i = b \circ a$. The maps λ and b are maps from X and B to D that agree when restricted to A, so by the universal property of the pushout there exists a morphism μ such that $\mu \circ j = b$ and $\mu \circ x = \lambda$. It remains to show that $p \circ \mu = y$. But $p \circ \mu$ and y are maps from the pushout Y to E such that $(p \circ \mu)j = yj$ and $(p \circ \mu)x = yx$. By the uniqueness part of the universal property of the pushout it follows that $p \circ \mu = y$. $\square$

The second step is the following proposition.

Proposition 8.13. *For every n, the inclusion morphism $S(n) \to D(n-1)$ is a cofibration.*

Proof. Recall that $S(n) = (\Lambda x, dx = 0)$ and $D(n-1) = (\Lambda(x, y), dx = 0, dy = x)$, where $|x| = n$ and $|y| = n - 1$. Consider the lifting problem

$$
\begin{tikzcd}
S(n) \arrow[r, "f"] \arrow[d] & B \arrow[d, two heads, "\pi"', "\sim"] \\
D(n-1) \arrow[r, "g"'] \arrow[ur, dashed, "\lambda"] & C
\end{tikzcd}
$$

Since the underlying algebras of $S(n)$ and $D(n-1)$ are free, in order to construct the lift λ, we only need to specify elements $\lambda(x) \in B^n$ and $\lambda(y) \in B^{n-1}$ and check that

$$
\begin{aligned}
d\lambda(x) &= 0, \\
\lambda(x) &= f(x), \\
d\lambda(y) &= f(x), \\
\pi\lambda(y) &= g(y).
\end{aligned}
$$

We obviously have no choice but to set $\lambda(x) := f(x)$. The element $f(x)$ is a cocycle because $df(x) = f(dx) = 0$. It remains to find an element $\lambda(y)$ with the desired properties. By commutativity of the square, $\pi f(x) = g(x) = g(dy) = d(gy)$, so $[f(x)] \in \ker \mathrm{H}^n(\pi)$. But π is a quasi-isomorphism, so $[f(x)] = 0$, i.e., $f(x) = d(b)$ for some $b \in B^{n-1}$. The difference $\pi(b) - g(y)$ is a cocycle, because $d\pi(b) = \pi(db) = \pi f(x) = g(dy) = dg(y)$. Since $\mathrm{H}^{n-1}(\pi)$ is surjective, we can find a cocycle $b' \in B^{n-1}$ and a cochain $c \in C^{n-2}$ such that $\pi(b') = \pi(b) - g(y) + d(c)$. Since π is surjective, we can find $b'' \in B^{n-2}$ with $\pi(b'') = c$. Finally, if we define

$$\lambda(y) := b - b' + db'',$$

then one checks that $\pi\lambda(y) = g(y)$ and $d\lambda(y) = f(x)$, as required. $\square$

Proof of Theorem 8.11. This follows from the definition of relative Sullivan algebras, by applying Proposition 8.12 and Proposition 8.13. □

Factorizations

Proposition 8.14. *Every morphism of cochain algebras $f\colon A \to B$ can be factored as*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "j"', "\sim"] & & B \\ & Y \arrow[ur, "p"'] & \end{tikzcd}$$

where p is surjective and j is a split cofibration and a quasi-isomorphism.

Proof. Let $V \subseteq B$ be a graded vector space that generates B as an algebra. The inclusion $V \subseteq B$ extends to a surjective morphism of cochain algebras $q\colon D(s^{-1}V) \to B$. We can take

$$A \xrightarrow{j} A \otimes D(s^{-1}V) \xrightarrow{p} B$$

as our factorization, where j is the inclusion $a \mapsto a \otimes 1$, and p is defined by $a \otimes x \mapsto f(a)q(x)$. Indeed, it is clear that $(A \otimes D(s^{-1}V), A)$ is a relative Sullivan algebra, so j is a cofibration. Morover, if define $\sigma\colon A \otimes D(s^{-1}V) \to A$ by $\sigma(a \otimes x) = \epsilon(x)a$, where $\epsilon\colon D(s^{-1}V) \to \Bbbk$ is the augmentation sending $s^{-1}V$ to zero, then $\sigma \circ j = 1_A$, so that j is split. Finally, since $\Bbbk \to D(s^{-1}V)$ is a quasi-isomorphism, it follows that j is a quasi-isomorphism. □

8.2 Mapping spaces and homotopy

In this section, we will introduce the notion of homotopy between morphisms of cochain algebras, prove the uniqueness of Sullivan models up to homotopy equivalence, and prove a cochain algebra version of the Whitehead theorem.

The set of path components of a simplicial set

Let X be a simplicial set. Define a relation $\simeq$ on the set of 0-simplices X_0 by declaring that $f \simeq g$, for $f, g \in X_0$, if there is a 1-simplex $h \in X_1$ such that $d_0(h) = g$ and $d_1(h) = f$.

$$\bullet_f \xrightarrow{h} \bullet_g$$

In general, the relation $\simeq$ is not an equivalence relation, but when it is we can form the quotient

$$\pi_0(X) = X_0/\simeq .$$

This set is called the set of path componets of X.

Simplicial mapping spaces for topological spaces

Let U and T be topological spaces. The (simplicial) mapping space $\mathrm{map}(U, T)$ is defined to be the simplicial set with n-simplices

$$\mathrm{map}(U, V)_n = \mathrm{Hom}_{\mathbf{Top}}(U \times \Delta^n, T),$$

the set of continuous maps $U \times \Delta^n \to T$, where Δ^n is the standard topological n-simplex. If $\varphi\colon [m] \to [n]$ is a map in Δ, then

$$\varphi^*\colon \operatorname{map}(U,T)_n \to \operatorname{map}(U,T)_m$$

is defined by $\varphi^*(f) = f \circ (1 \times \varphi_*)$. Note that if we take $U = *$, then

$$\operatorname{map}(*,T) = S_\bullet(T),$$

the singular simplicial set associated to T.

A 1-simplex of $\operatorname{map}(U,T)$ is a continuous map $h\colon U \times \Delta^1 \to T$, and its 0-faces are the continuous maps

$$d_0(h), d_1(h)\colon U \to T$$

given by $d_0(h)(u) = h(u,1)$ and $d_1(h)(u) = h(u,0)$. Therefore, for 0-simplices f, g in $\operatorname{map}(U,T)$, i.e., continuous maps $f, g\colon U \to T$, we have that $f \simeq g$ as 0-simplices of the simplicial set $\operatorname{map}(U,T)$ if and only if f is homotopic to g in the usual sense. As should be familiar, homotopy between continuous maps is an equivalence relation, and we can identify the set of path components of the simplicial set $\operatorname{map}(U,T)$ with the set of homotopy classes of continuous maps from U to T,

$$\pi_0 \operatorname{map}(U,T) = [U,T].$$

The notion of homotopy relative to a subspace can also be interpreted in the simplicial language. Recall that if $V \subset U$ is a subspace, and if we are given a map $k\colon V \to T$, then two maps $f, g\colon U \to T$ with $f|_V = g|_V = k$ are said to be homotopic rel. V if there is a map $h\colon U \times \Delta^1 \to T$ such that $h(u,0) = f(u)$, $h(u,1) = g(u)$ for all $u \in U$ and $h(v,t) = k(v)$ for all $v \in V$ and all $t \in \Delta^1$.

Given a map $k\colon V \to T$, we can define the relative mapping space $\operatorname{map}_V(U,T)$ to be the pullback

$$\begin{tikzcd}
\operatorname{map}_V(U,T) \arrow[r] \arrow[d] & \operatorname{map}(U,T) \arrow[d, "(-)|_V"] \\
* \arrow[r, "k"] & \operatorname{map}(V,T)
\end{tikzcd}$$

Exercise 8.15. Check that the set of path components $\pi_0(\operatorname{map}_V(U,T))$ may be identified with the set $[U,T]_V$ of homotopy classes of maps rel. V.

Simplicial mapping spaces for cochain algebras

Definition 8.16. Let X and B be cochain algebras. The simplicial mapping space $\operatorname{map}(X,B)$ is defined to be the simplicial set with n-simplices

$$\operatorname{map}(X,B)_n = \operatorname{Hom}_{dga}(X, B \otimes \Omega_n^*),$$

the set of morphisms of cochain algebras $f\colon X \to B \otimes \Omega_n^*$, where $\Omega_\bullet^*$ is the simplicial de Rham algebra. For a morphisms $\varphi\colon [m] \to [n]$ in Δ, the map

$$\varphi^*\colon \operatorname{map}(X,B)_n \to \operatorname{map}(X,B)_m$$

is defined by $\varphi^*(f) = (1_B \otimes \varphi^*) \circ f$.

There is an isomorphism

$$\begin{tikzcd} \Omega_1^* \arrow[r, "\cong"] \arrow[d, shift right, "\partial_1"'] \arrow[d, shift left, "\partial_0"] & \Lambda(t,dt) \arrow[d, shift right, "\epsilon_1"'] \arrow[d, shift left, "\epsilon_0"] \\ \Omega_0^* \arrow[r, "\cong"] & \mathbb{k} \end{tikzcd}$$

where ϵ_0 and ϵ_1 are the cochain algebra morphisms determined by $\epsilon_0(t) = 0$ and $\epsilon_1(t) = 1$, respectively. In particular, the set of 0-simplices of the simplicial set $\mathrm{map}(X,B)$ may be identified with the set of morphisms of cochain algebras from X to B. It also leads us to the following definition of homotopy between morphisms.

Definition 8.17. Two morphisms of cochain algebras $f, g\colon X \to B$ are *homotopic* if there is a morphism of cochain algebras

$$h\colon X \to B \otimes \Lambda(t,dt)$$

such that $h|_{t=0} = g$ and $h|_{t=1} = f$.

We now face the problem that homotopy between morphisms of cochain algebras is not an equivalence relation in general. However, we have the following theorem.

Theorem 8.18. *If X is a cofibrant cochain algebra, then homotopy between morphisms $X \to B$ is an equivalence relation.*

This could be proved directly, but we prefer to derive this as a corollary to a much more general theorem that we will prove later, Theorem 8.31.

We will also need the notion of relative homotopy.

Definition 8.19. Let $i\colon A \to X$ and $k\colon A \to B$ be morphisms of cochain algebras. Given two morphisms $f, g\colon X \to B$ such that $f \circ i = g \circ i = k$, we say that f is *homotopic to g relative to A*, written $f \simeq g$ rel. A, if there is a morphism of cochain algebras

$$h\colon X \to B \otimes \Lambda(t,dt)$$

such that $h|_{t=0} = g$ and $h|_{t=1} = f$, and such that the restriction

$$h \circ i\colon A \to B \otimes \Lambda(t,dt)$$

is the 'constant homotopy' $a \mapsto k(a) \otimes 1$.

As in the case of topological spaces, relative homotopy can be interpreted simplicially. We define the relative mapping space $\mathrm{map}_A(X,B)$ by declaring the following diagram to be a pullback

$$\begin{tikzcd} \mathrm{map}_A(X,B) \arrow[r] \arrow[d] & \mathrm{map}(X,B) \arrow[d, "i^*"] \\ * \arrow[r, "k"] & \mathrm{map}(A,B) \end{tikzcd} \tag{13}$$

The 0-simplices of $\mathrm{map}_A(X,B)$ are the morphisms $f\colon X \to B$ such that $f \circ i = k$, and a 1-simplex $h \in \mathrm{map}_A(X,B)_1$ with 0-faces $d_0(h) = f$ and $d_1(h) = g$ is exactly a homotopy $h\colon f \simeq g$ rel. A as in Definition 8.19

Theorem 8.20 ('Lifting theorem')**.** *Let $i\colon A \to X$ be a cofibration of cochain algebras. In every commutative square of cochain algebras*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "\pi", "\sim"'] \\ X \arrow[r, "g"] \arrow[ur, dashed, "\lambda"] & C \end{tikzcd}$$

where π is a quasi-isomorphism, there exists a morphism $\lambda\colon X \to B$ such that

- $\lambda \circ i = f$
- $\pi \circ \lambda \simeq g$ *rel. A.*

Moreover, any two such lifts λ are homotopic relative to A.

This will also be derived as a consequence of Theorem 8.31. As a corollary, we obtain the uniqueness of Sullivan models up to homotopy equivalence.

Corollary 8.21. *Any two Sullivan models of a morphism of cochain algebras $f\colon A \to B$ are homotopy equivalent relative to A.*

Proof. Let (X, A) and (Y, A) be two Sullivan models for f. Then we can apply the Lifting theorem to the following diagram four times:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d, "\sim"] \arrow[dl, dashed, shift left, "\mu", "\sim"'] \\ X \arrow[r, "\sim"'] \arrow[ur, dashed, shift left, "\lambda"] & B \end{tikzcd}$$

We apply the existence part twice to deduce the existence of λ and μ, and we use the uniqueness up to homotopy twice to deduce $\lambda \circ \mu \simeq 1_Y$ rel. A and $\mu \circ \lambda \simeq 1_X$ rel. A. ☐

Interlude: Kan fibrations

We will need some more simplicial homotopy theory. The reader should consult [8] or [17] for proofs.

Definition 8.22. Let $p\colon X \to Y$ and $i\colon A \to B$ be morphisms in some category $\mathcal{C}$. We will say that 'p has the right lifting property with respect to i' or 'i has the left lifting property with respect to p' if in every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "\lambda"] & Y \end{tikzcd}$$

there exists a morphism $\lambda\colon B \to X$ making both triangles commute.

For $n \geq 1$ and $0 \leq k \leq n$, let $\Lambda^k[n] \subset \Delta[n]$ denote the simplicial set associated to the abstract simplicial complex $2^{[n]} \setminus \{[n], [n]\setminus\{k\}\}$. The simplicial set $\Lambda^k[n]$ is called the 'k^{th} horn'. The terminology and the notation is explained by the following picture of $\Lambda^1[2]$:

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

Its underlying abstract simplicial complex is $\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}\}$.

Definition 8.23. A simplicial map $f\colon X \to Y$ is called a *weak equivalence* if the induced map on geometric realizations $|f|\colon |X| \to |Y|$ is a weak homotopy equivalence, i.e., for every $x \in |X|$ and every $n \geq 0$, the map induced by f,

$$\pi_n(|X|, x) \to \pi_n(Y, |f|(x)),$$

is a bijection.

Theorem 8.24. *The following are equivalent for a simplicial map $p\colon X \to Y$.*

1. *The map p has the right lifting property with respect to all inclusion maps $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$.*
2. *The map p has the right lifting property with respect to all inclusion maps of simplicial sets $i\colon L \hookrightarrow K$ that are weak equivalences.*

Definition 8.25. A simplicial map $p\colon X \to Y$ is called a *Kan fibration* if the equivalent conditions in Theorem 8.24 are fulfilled. A simplicial set X is called a *Kan complex* if the map $X \to *$ is a Kan fibration.

Theorem 8.26. *The following are equivalent for a simplicial map $p\colon X \to Y$.*

1. *The map p is both a Kan fibration and a weak equivalence.*
2. *The map p has the right lifting property with respect to all inclusion maps $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq 0$.*
3. *The map p has the right lifting property with respect to all inclusion maps of simplicial sets $i\colon L \hookrightarrow K$.*

Definition 8.27. A simplicial map $p\colon X \to Y$ is called a *trivial Kan fibration* if the equivalent conditions in Theorem 8.24 are fulfilled. A simplicial set X is called a *trivial Kan complex* if the map $X \to *$ is a Kan fibration.

Proposition 8.28. *If X is a Kan complex, then the homotopy relation $\simeq$ is an equivalence relation on the set of 0-simplices X_0.*

Proof. (Reflexivity): Let $f \in X_0$. The degenerate 1-simplex $h = s_0(f)$ is a homotopy $h\colon f \simeq f$, because of the simplicial identities $d_0 s_0 = d_1 s_0 = 1$.

(Transitivity): Let $h_{01}\colon f_0 \simeq f_1$ and $h_{12}\colon f_1 \simeq f_2$. This data determines a simplicial map $\tau\colon \Lambda^1[2] \to X$.

$$\begin{tikzcd} & f_1 \arrow[dr, "h_{01}"] & \\ f_0 \arrow[ur, "h_{12}"] \arrow[rr, dashed, "d_1(\sigma)"'] & & f_2 \end{tikzcd}$$

Since X is a Kan complex, we can extend τ to a map $\sigma\colon \Delta[2] \to X$. The first face $d_1(\sigma)$ is then a homotopy $f_0 \simeq f_2$.

(Symmetry): Let $h\colon f \simeq g$. By using the trivial homotopy $s_0(f)\colon f \simeq f$, we get a map $\Lambda^0[2] \to X$ as indicated below. Since X is Kan, this extends to a map $\sigma\colon \Delta[2] \to X$, and $d_0(\sigma)$ is a homotopy $g \simeq f$.

$$\begin{tikzcd} & g \arrow[dr, dashed, "d_0(\sigma)"] & \\ f \arrow[ur, "h"] \arrow[rr, "s_0(f)"'] & & f \end{tikzcd}$$

□

Definition 8.29. A simplicial map $p\colon X \to Y$ is called a *rational Kan fibration* if it has the right lifting property with respect to all inclusion maps of simplicial sets $i\colon L \hookrightarrow K$ that induce an isomorphism in rational homology

$$i_*\colon \mathrm{H}_n(L;\mathbb{Q}) \xrightarrow{\cong} \mathrm{H}_n(K;\mathbb{Q})$$

for all $n \geq 0$.

A simplicial set X is called a *rational Kan complex* if the map $X \to *$ is a rational Kan fibration.

Clearly, every rational Kan fibration is a Kan fibration, because every weak homotopy equivalence induces an isomorphism in rational homology. In particular, every rational Kan complex is a Kan complex.

The following proposition is a consequence of the fact that (trivial/rational) Kan fibrations are characterized by right lifting properties. The proof is dual to the proof of the second statement of Proposition 8.12.

Proposition 8.30. *Given a pullback diagram of simplicial sets,*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

if p is a (trivial/rational) Kan fibration, then so is p'.

The fundamental theorem

Given morphisms of cochain algebras $i\colon A \to X$ and $\pi\colon B \to C$, the commutativity of the diagram

$$\begin{tikzcd} \mathrm{map}(X,B) \arrow[r, "\pi_*"] \arrow[d, "i^*"'] & \mathrm{map}(X,C) \arrow[d, "i^*"] \\ \mathrm{map}(A,B) \arrow[r, "\pi_*"] & \mathrm{map}(A,C) \end{tikzcd}$$

yields a map

$$i * \pi\colon \mathrm{map}(X,B) \to \mathrm{map}(A,B) \times_{\mathrm{map}(A,C)} \mathrm{map}(X,C).$$

The following is the fundamental theorem about simplicial mapping spaces.

Theorem 8.31. *Let $i\colon A \to X$ be a cofibration and $\pi\colon B \to C$ a surjection of cochain algebras. Then the simplicial map*

$$i * \pi\colon \operatorname{map}(X,B) \to \operatorname{map}(A,B) \times_{\operatorname{map}(A,C)} \operatorname{map}(X,C)$$

is a rational Kan fibration. If, in addition, i or π is a quasi-isomorphism, then $i * \pi$ is a weak equivalence.

The proof will be given in the next section. First we will derive some important consequences.

Corollary 8.32. *1. If $i\colon A \to X$ is a cofibration between cochain algebras, then $i^*\colon \operatorname{map}(X,B) \to \operatorname{map}(A,B)$ is a rational Kan fibration.*

2. If $\pi\colon B \to C$ is a surjective morphism of cochain algebras, and if X is cofibrant, then $\pi_*\colon \operatorname{map}(X,B) \to \operatorname{map}(X,C)$ is a rational Kan fibration.

Proof. For the first statement, take $C = 0$ in Theorem 8.31. For the second statement, take $A = \Bbbk$. □

Corollary 8.33. *Let $i\colon A \to X$ and $k\colon A \to B$ be morphisms of cochain algebras. If i is a cofibration, then the relative mapping space $\operatorname{map}_A(X,B)$ is a rational Kan complex. In particular, homotopy rel. A is an equivalence relation on the set of morphisms of cochain algebras from X to B under A.*

Proof. Consider the pullback diagram (13) defining the relative mapping space $\operatorname{map}_A(X,B)$. By the previous corollary, the map $i^*\colon \operatorname{map}(X,B) \to \operatorname{map}(A,B)$ is a rational Kan fibration. Hence, by Proposition 8.30, the map $\operatorname{map}_A(X,B) \to *$ is a rational Kan fibration. Every rational Kan complex is a Kan complex, so the statement about homotopy rel. A follows from Proposition 8.28 □

Proposition 8.34. *Let $i\colon A \to X$ be a cofibration. Given a quasi-isomorphism $\pi\colon B \to C$ of cochain algebras, the induced map*

$$\pi_*\colon [X,B]_A \to [X,C]_A$$

is a bijection.

Proof. If π is surjective: Consider the diagram

$$\begin{tikzcd}
\operatorname{map}_A(X,B) \arrow[r] \arrow[d, dashed] & \operatorname{map}(X,B) \arrow[d, "i*\pi"] & \\
\operatorname{map}_A(X,C) \arrow[r, dashed] \arrow[d] & P \arrow[r] \arrow[d] & \operatorname{map}(X,C) \arrow[d, "i^*"] \\
* \arrow[r] & \operatorname{map}(A,B) \arrow[r, "\pi_*"'] & \operatorname{map}(A,C).
\end{tikzcd}$$

The right square is the pullback defining $P := \operatorname{map}(A,B) \times_{\operatorname{map}(A,C)} \operatorname{map}(X,C)$. The two rectangles in the diagram are the pullbacks defining the relative mapping spaces $\operatorname{map}_A(X,B)$ and $\operatorname{map}_A(X,C)$. By general pullback yoga, it follows that the two remaining squares are pullbacks. Since i is a cofibration and π is a surjective quasi-isomorphism, Theorem 8.31 says that $i * \pi$ is a rational Kan fibration and a weak equivalence. In particular, the induced map

$$\pi_*\colon \pi_0(\mathrm{map}_A(X,B)) \to \pi_0(\mathrm{map}_A(X,C))$$

is a bijection.

For arbitrary π: As in Proposition 8.14 we can factor π as $\pi = p \circ j$,

$$\begin{tikzcd} B \arrow[rr, "\sim", "\pi"'] \arrow[dr, "\sim", "j"'] & & C \\ & Y \arrow[ur, "\therefore\sim"', "p"] & \end{tikzcd}$$

where j is a split cofibration and a quasi-isomorphism, and p is surjective. Since π is a quasi-isomorphism, it follows that also p is a quasi-isomorphism. Hence, p_* is a bijection by the first part of the proof. By commutativity of the diagram

$$\begin{tikzcd} {[X,B]_A} \arrow[rr, "\pi_*"'] \arrow[dr, "j_*"'] & & {[X,C]_A} \\ & {[X,Y]_A} \arrow[ur, "\cong", "p_*"'] & \end{tikzcd}$$

we are done if we can prove that j_* is a bijection. Let $\sigma\colon Y \to B$ be the splitting of j, i.e., $\sigma \circ j = 1_B$. Then σ is necessarily a surjective quasi-isomorphism. By the first part of the proof, $\sigma_*\colon [X,Y]_A \to [X,B]_A$ is a bijection. Since $\sigma_* \circ j_* = 1$, it follows that j_* is the inverse of σ_*, and in particular j_* itself is a bijection. $\square$

The proof of the Lifting theorem for cochain algebras, Theorem 8.20, is now complete.

Proof of Theorem 8.31

In view of the characterization of trivial and rational Kan fibrations in terms of lifting properties (see Theorem 8.26 and Definition 8.29), Theorem 8.31 can be reformulated as follows.

Theorem 8.35. *Let $i\colon A \to X$ be a cofibration and $\pi\colon B \to C$ a surjection of cochain algebras, and let $j\colon L \to K$ be an inclusion of finite simplicial sets. The following lifting problem in the category of simplicial sets*

$$\begin{tikzcd} L \arrow[r] \arrow[d, "j"'] & \mathrm{map}(X,B) \arrow[d, "i*\pi"] \\ K \arrow[r] \arrow[ur, dashed] & \mathrm{map}(A,B) \times_{\mathrm{map}(A,C)} \mathrm{map}(X,C) \end{tikzcd} \tag{14}$$

can be solved if p is a quasi-isomorphism, or i is a quasi-isomorphism, or j induces an isomorphism in rational homology.

Proposition 8.36. *Let A and B be cochain algebras, and let K be a simplicial set. There is a natural simplicial map*

$$\psi\colon \mathrm{Hom}_{dga}(A, B \otimes \Omega^*(K)) \to \mathrm{Hom}_{s\mathbf{Set}}(K, \mathrm{map}(A,B)).$$

If K has finitely many non-degenerate simplices, or if B is of finite type, then ψ is a bijection.

Proof. Let $f\colon A \to B \otimes \Omega^*(K)$ be a morphism of cochain algebras. We define

$$\psi(f)_n\colon K_n \to \mathrm{map}(A,B)_n = \mathrm{Hom}_{dga}(A, B \otimes \Omega^*_n)$$

as follows: Given an n-simplex $x \in K_n$, thought of as a simplicial map $x\colon \Delta[n] \to K$, there is an induced morphism of cochain algebras

$$x^*\colon \Omega^*(K) \to \Omega^*(\Delta[n]) = \Omega^*_n.$$

We define $\psi(f)(x)$ to be the composite morphism of cochain algebras

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\psi(f)(x)"'] & B \otimes \Omega^*(K) \arrow[d, "1\otimes x^*"] \\ & B \otimes \Omega^*_n. \end{tikzcd}$$

It is obvious that $\psi(f)$ is a simplicial map, and that ψ is natural in A, B and K.

The map ψ is evidently a bijection for $K = \Delta[n]$. Since we work over a field, all modules are flat and the tensor product functor $B \otimes -$ preserves pullbacks. Therefore, both the source and target of ψ take pushouts to pullbacks (in the K variable). If K has finitely many non-degenerate simplices, then, as can be seen by using the skeletal filtration, K is built from standard simplices by taking iterated pushouts, so it follows that ψ is an isomorphism for all such K.

On the other hand, if B is of finite type, i.e., B^n is finite dimensional for every n, then the functor $B\otimes-$ commutes with arbitrary limits. In this case, both the source and target of ψ take arbitrary colimits to limits. Since every simplicial set is a colimit of standard simplices, it follows that ψ is an isomorphism for all simplicial sets K in this case. □

Exercise 8.37. Use Proposition 8.36 to prove that the lifting problem (14) is equivalent to the following lifting problem in the category of cochain algebras

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \otimes \Omega^*(K) \arrow[d, "\pi * \Omega^*(j)"] \\ X \arrow[r] \arrow[ur, dashed] & B \otimes \Omega^*(L) \times_{C\otimes\Omega^*(L)} C \otimes \Omega^*(K). \end{tikzcd}$$

Proposition 8.38. *If $p\colon B \to C$ and $q\colon D \to E$ are surjective morphisms of cochain algebras, then the induced morphism*

$$p * q\colon B \otimes D \to B \otimes E \times_{C\otimes E} C \otimes D$$

is surjective. If in addition p or q is a quasi-isomorphism, then so is $p * q$.

Proof. Let $I = \ker(p)$ and $P = B\otimes E \times_{C\otimes E} C\otimes D$. Since $\Bbbk$ is a field, the functor $A \otimes -$ is exact and preserves quasi-isomorphisms for every cochain complex A. Since p is surjective, and since surjections are stable under pullbacks, there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \otimes E \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & C \otimes D \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \otimes E \arrow[r] & B \otimes E \arrow[r] & C \otimes E \arrow[r] & 0 \end{tikzcd}$$

where we are allowed to equate the kernels because the right square is a pullback. This explains the existence and exactness of the bottom row in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & I \otimes D \arrow[r] \arrow[d, "1\otimes q"] & B \otimes D \arrow[r] \arrow[d, "p*q"] & C \otimes D \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \otimes E \arrow[r] & P \arrow[r] & C \otimes D \arrow[r] & 0
\end{tikzcd}$$

The left vertical arrow $1 \otimes q$ is surjective because q is surjective. It then follows from the snake lemma that $p * q$ is surjective. If q is a quasi-isomorphism, then so is $1 \otimes q$, and it follows from the five lemma that $p * q$ is as well. A symmetric argument shows that $p * q$ is a quasi-isomorphism if p is. $\square$

We can now complete the proof of Theorem 8.31.

Proposition 8.39. *Let $f, g\colon A \to B$ be morphisms of cochain algebras. If f and g are homotopic as morphisms of cochain algebras, then they are homotopic as morphisms of cochain complexes. In particular, $\mathrm{H}^n(f) = \mathrm{H}^n(g)$ for all n.*

Proof. Let $h\colon A \to B \otimes \Lambda(t, dt)$ be a homotopy from f to g, i.e., a morphism of cochain algebras such that $\epsilon_0 \circ h = g$ and $\epsilon_1 \circ h = f$, where $\epsilon_0, \epsilon_1\colon B \otimes \Lambda(t, dt) \to B$ are the morphisms of cochain algebras determined by $\epsilon_1(1 \otimes t) = 1$, $\epsilon_0(1 \otimes t) = 0$ and $\epsilon_i(b \otimes 1) = b$. Let $I \subset \Lambda(t, dt)$ be the dg ideal generated by $t(t-1)$. The morphisms ϵ_0 and ϵ_1 factor as

$$\begin{tikzcd}
B \otimes \Lambda(t, dt) \arrow[rr, "\epsilon_i"] \arrow[dr, "p"'] & & B \\
& B \otimes \Lambda(t, dt)/I \arrow[ur, "\overline{\epsilon}_i"'] &
\end{tikzcd}$$

where p is the quotient map. We will prove that the morphisms $\overline{\epsilon}_0$ and $\overline{\epsilon}_1$ are chain homotopic. Then it will follow that $g = \overline{\epsilon}_0 ph$ is chain homotopic to $f = \overline{\epsilon}_1 ph$. As a graded vector space, the quotient $\Lambda(t, dt)/I$ has basis $1-t$, t, dt. Hence, an n-cochain b of $B \otimes \Lambda(t, dt)/I$ can be written uniquely as

$$b = b_0(1-t) + b_1 t + c dt$$

where $b_0, b_1 \in B^n$ and $c \in B^{n-1}$. Clearly, $\overline{\epsilon}_i(b) = b_i$ for $i = 0, 1$. Let us define a chain homotopy $k\colon B \otimes \Lambda(t, dt) \to B$ by

$$k(b) := (-1)^n c,$$

for b an n-cochain as above. We claim that

$$\overline{\epsilon}_0 - \overline{\epsilon}_1 = kd + dk.$$

We leave the verification to the reader. $\square$

8.3 Spatial realization

Definition 8.40. Let A be a commutative cochain algebra. The *spatial realization* of A is the simplicial set

$$\langle A \rangle := \mathrm{map}(A, \Bbbk) = \mathrm{Hom}_{dga}(A, \Omega^*_\bullet).$$

Exercise 8.41. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant adjunction* between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of contravariant functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

together with a natural bijection

$$\psi\colon \operatorname{Hom}_{\mathcal{D}}(X, FY) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{C}}(Y, GX)$$

for objects X in $\mathcal{D}$ and Y in $\mathcal{C}$.

1. Given a contravariant adjunction as above, let $\eta_Y := \psi(1_{FY})\colon Y \to GFY$. For a morphism $f\colon X \to GY$ in $\mathcal{C}$, prove that the diagram

$$\begin{tikzcd} Y \arrow[rr, "\psi(f)"] \arrow[dr, "\eta_Y"'] & & GX \\ & GFY \arrow[ur, "Gf"'] & \end{tikzcd}$$

   commutes.

2. Prove that F and G take arbitrary colimits to limits.

Proposition 8.36 implies (plug in $B = \Bbbk$) that the de Rham algebra functor $\Omega^*(-)$ and the spatial realization functor $\langle - \rangle$ are part of a contravariant adjunction

$$\begin{tikzcd} s\mathbf{Set} \arrow[r, shift left, "\Omega^*(-)"] & \mathbf{CDGA}. \arrow[l, shift left, "\langle - \rangle"] \end{tikzcd} \tag{15}$$

In other words, there is a natural bijection

$$\psi\colon \operatorname{Hom}_{dga}(A, \Omega^*(K)) \xrightarrow{\cong} \operatorname{Hom}_{s\mathbf{Set}}(K, \langle A \rangle). \tag{16}$$

for commutative cochain algebras A and simplicial sets K. We note the following properties of this adjunction:

1. The contravariant functor $\Omega^*(-)\colon s\mathbf{Set} \to \mathbf{CDGA}$ takes rational homotopy equivalences to quasi-isomorphisms, and it takes inclusions of simplicial sets to surjections of cochain algebras.

2. The contravariant functor $\langle - \rangle\colon \mathbf{CDGA} \to s\mathbf{Set}$ takes cofibrations of cochain algebras to rational Kan fibrations and it takes trivial cofibrations to trivial fibrations.

Next, we will investigate to what extent homotopies are preserved by the functors $\Omega^*(-)$, $\langle - \rangle$, and by the bijection (16).

Lemma 8.42. *Let $f, g\colon A \to B$ be morphisms of cochain algebras. If f and g are homotopic, then the simplicial maps $\langle f \rangle, \langle g \rangle\colon \langle B \rangle \to \langle A \rangle$ are homotopic.*

Proof. The isomorphism $\Lambda(t,dt) \xrightarrow{\cong} \Omega^*(\Delta[1])$ corresponds to a map $j\colon \Delta[1] \to \langle \Lambda(t,dt)\rangle$ under the bijection (16). If $h\colon A \to B \otimes \Lambda(t,dt)$ is a homotopy between f and g, then we get a simplicial homotopy $\langle B\rangle \times \Delta[1] \to \langle A\rangle$ between $\langle f\rangle$ and $\langle g\rangle$ by taking the following composite:

$$\langle B\rangle \times \Delta[1] \xrightarrow{1\times j} \langle B\rangle \times \langle \Lambda(t,dt)\rangle \xrightarrow{\cong} \langle B \otimes \Lambda(t,dt)\rangle \xrightarrow{\langle h\rangle} \langle A\rangle.$$

The isomorphism in the middle comes from the fact that $\langle - \rangle$ takes coproducts to products. $\square$

Unfortunately, the de Rham algebra functor $\Omega^*(-)$ does not preserve homotopies, but it does so 'in the eyes of cofibrant cochain algebras' in the following sense.

Lemma 8.43. *Let $f, g\colon K \to L$ be maps of simplicial sets. If f and g are homotopic, then there is an equality of the induced maps*

$$f^* = g^*\colon [A, \Omega^*(L)] \to [A, \Omega^*(K)],$$

for every cofibrant commutative cochain algebra A.

Proof. That f is homotopic to g means that there is a commutative diagram of simplicial maps

$$\begin{tikzcd} K \arrow[d, "d^0"'] \arrow[dr, "f"] & \\ K \times \Delta[1] \arrow[r, "h"] & L \\ K \arrow[u, "d^1"] \arrow[ur, "g"'] & \end{tikzcd}$$

If we apply the functor $[A, \Omega^*(-)]$ to this diagram we get

$$\begin{tikzcd} {[A, \Omega^*(K)} & \\ {[A, \Omega^*(K \times \Delta[1])]} \arrow[u, "(d^0)^*"] \arrow[d, "(d^1)^*"'] & {[A, \Omega^*(L)]} \arrow[l, "h^*"] \arrow[ul, "f^*"'] \arrow[dl, "g^*"] \\ {[A, \Omega^*(K)]} & \end{tikzcd}$$

If $p\colon K \times \Delta[1] \to K$ denotes the projection, then clearly $pd^0 = pd^1$. Since p is a weak equivalence and A is cofibrant, the map

$$p^*\colon [A, \Omega^*(K)] \to [A, \Omega^*(K \times \Delta[1])]$$

is a bijection. In particular, we can cancel p^* in the equality $(d^0)^*p^* = (d^1)^*p^*$ and conclude that $(d^0)^* = (d^1)^*$. Hence, $f^* = (d^0)^*h^* = (d^1)^*h^* = g^*$. $\square$

Proposition 8.44. *The bijection* (16) *induces a bijection*

$$[A, \Omega^*(K)] \cong [K, \langle A\rangle]$$

for cofibrant cochain algebras A and arbitrary simplicial sets K.

Proof. It suffices to check that, in (16), ψ and ψ^{-1} preserve homotopies. Let us begin by checking that ψ preserves homotopies. Let $f\colon A \to \Omega^*(K)$ be a morphism of cochain algebras. By Exercise 8.41, there is a factorization of $\psi(f)$ as

$$\begin{tikzcd} K \arrow[rr, "\psi(f)"] \arrow[dr, "\eta_K"'] & & \langle A\rangle \\ & \langle \Omega^*(K)\rangle \arrow[ur, "\langle f\rangle"'] & \end{tikzcd}$$

Therefore, it suffices to prove that $\langle - \rangle$ preserves homotopies. But this was done in Lemma 8.42. Similarly, since A is assumed to be cofibrant, the corresponding factorization of $\psi^{-1}(g)$ together with Lemma 8.43 proves that ψ^{-1} preserves homotopies. □

Corollary 8.45. *For $n \geq 2$, $\langle S(n)\rangle \simeq K(\Bbbk, n)$.*

Proof. If A is a cochain algebra with $A^0 \cong \Bbbk$ and $A^k = 0$ for $k < n$, then the simplicial set $\langle A\rangle$ has a single k-simplex for $k = 0, 1, \ldots, n-1$. This follows from the fact that the cochain algebra Ω_k^* is zero in degrees $* > k$. Hence, the simplicial set $\langle S(n)\rangle$ has a unique k-simplex for $k < n$, and is in particular n-connected. By Proposition 8.44,

$$\pi_k\langle S(n)\rangle \cong [S^k, \langle S(n)\rangle] \cong [S(n), \Omega^*(S^k)] \cong \mathrm{H}^n(S^k; \Bbbk) = \begin{cases} \Bbbk, & k = n, \\ 0, & k \neq n. \end{cases}$$

This proves the claim. □

Theorem 8.46. *Let $\Bbbk = \mathbb{Q}$, and let A be a cofibrant commutative cochain algebra such that $\mathrm{H}^0(A) = \mathbb{Q}$, $\mathrm{H}^1(A) = 0$ and $\mathrm{H}^p(A)$ is finite dimensional for every p. Then the map*

$$\eta_A\colon A \to \Omega^*\langle A\rangle$$

is a quasi-isomorphism.

The proof of this theorem will be given in the next section.

Corollary 8.47. *Let K be a simply connected simplicial set of finite $\mathbb{Q}$-type, and let $f\colon A \xrightarrow{\sim} \Omega^*(K)$ be a Sullivan model for K. Then the adjoint map $\psi(f)\colon K \to \langle A\rangle$ is a $\mathbb{Q}$-localization.*

Proof. We need to check that $\langle A\rangle$ is $\mathbb{Q}$-local and that $\psi(f)\colon K \to \langle A\rangle$ is a rational homotopy equivalence. Since A is cofibrant, the simplicial set $\langle A\rangle$ is a rational Kan complex, so it is $\mathbb{Q}$-local. Next, $\psi(f)\colon K \to \langle A\rangle$ is a rational homotopy equivalence if and only if $\Omega^*(\psi(f))$ is a quasi-isomorphism, and that this is the case follows from Theorem 8.46 together with commutativity of the diagram

$$\begin{tikzcd} \Omega^*\langle A\rangle \arrow[rr, "\Omega^*(\psi(f))"] & & \Omega^*(K) \\ & A \arrow[ul, "\sim", "\eta_A"'] \arrow[ur, "\sim"', "f"] & \end{tikzcd}$$

□

Theorem 8.48. *The spatial realization functor induces an equivalence of categories*

$$\langle - \rangle\colon \operatorname{Ho}(\mathbf{CDGA}_{\mathbb{Q}}^{cof,f,1}) \xrightarrow{\simeq} \operatorname{Ho}(s\mathbf{Set}_{\mathbb{Q}}^{f,1})$$

The left hand side denotes the homotopy category of cofibrant, simply connected, finite type, commutative cochain algebras. The right hand side denotes the homotopy category of simply connected rational Kan complexes of finite $\mathbb{Q}$-type.

Proof. We should first remark that the spatial realization functor induces a well-defined functor on homotopy categories because of Lemma 8.42.

Recall that a functor is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Fully faithfulness: Let A and B be cofibrant simply connected finite type commutative cochain algebras. We need to show that the induced map $[A, B] \to [\langle B\rangle, \langle A\rangle]$ is a bijection. This follows from the following factorization

$$\begin{tikzcd}
{[A,B]} \arrow[rr, "\therefore\cong"] \arrow[dr, "\cong", "(\eta_B)_*"'] & & {[\langle B\rangle, \langle A\rangle]} \\
& {[A, \Omega^*\langle B\rangle]} \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The left map, which is induced by the unit $\eta_B\colon B \to \Omega^*\langle B\rangle$, is a bijection because of two things: First, since B satisfies the hypotheses of Theorem 8.46, the map η_B is a quasi-isomorphism. Secondly, since A is cofibrant, the induced map $(\eta_B)_*$ is a bijection by Proposition 8.34. The right map is the bijection from Proposition 8.44.

Essential surjectivity: Every simply connected simplicial set K of finite $\mathbb{Q}$-type admits a Sullivan model $A \xrightarrow{\sim} \Omega^*(K)$. The adjoint $K \to \langle A\rangle$ is a $\mathbb{Q}$-localization by Corollary 8.47. If K is a rational Kan complex, then it is also $\mathbb{Q}$-local. Since every rational homotopy equivalence between $\mathbb{Q}$-local spaces is a homotopy equivalence, it follows that $K \to \langle A\rangle$ is a homotopy equivalence, i.e., an isomorphism in the homotopy category. This proves essential surjectivity. $\square$

The essential ingredients in proving Theorem 8.46 are the calculation of the rational cohomology of Eilenberg-Mac Lane spaces (Theorem 5.1) and the Eilenberg-Moore theorem, to be discussed next.

8.4 The Eilenberg-Moore theorem

The Eilenberg-Moore theorem addresses the problem of calculating the cohomology of homotopy fibers, or more generally homotopy pullbacks.

Theorem 8.49 (Eilenberg-Moore theorem)**.** *Let $\Bbbk$ be a field, and write $C^*(-) = C^*(-;\Bbbk)$ for brevity. Consider a pullback diagram of simplicial sets*

$$\begin{tikzcd}
X \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

where p is a Kan fibration. If

- *X, E and B are connected and of finite $\Bbbk$-type, and*
- *B is simply connected,*

then the induced morphism of cochain complexes

$$C^*(X) \otimes^{\mathbb{L}}_{C^*(B)} C^*(E) \to C^*(X \times_B E)$$

is a quasi-isomorphism. In particular, there is an isomorphism of graded vector spaces over $\Bbbk$

$$\mathrm{H}^*(X \times_B E) \cong \mathrm{Tor}_*^{C^*(B)}(C^*(X), C^*(E)).$$

A proof of the Eilenberg-Moore theorem can be found in [18, Theorem 7.14]. We need to explain what the derived tensor product $C^*(X) \otimes^{\mathbb{L}}_{C^*(B)} C^*(E)$ is.

For a right module M and a left module N over a differential graded algebra A, the derived tensor product $M \otimes^{\mathbb{L}}_A N$ is a certain cochain complex, functorial in each variable, equipped with a natural map to the ordinary tensor product $\epsilon\colon M \otimes^{\mathbb{L}}_A N \to M \otimes_A N$. It is characterized up to quasi-isomorphism by the following properties.

1. (Homotopy invariance) Given a morphism of dg-algebras $\varphi\colon A \to A'$ and morphisms of A-modules $f\colon M_A \to M'_{A'}$, $g\colon {}_AN \to {}_{A'}N'$, the induced morphism
$$f \otimes^{\mathbb{L}}_{\varphi} g\colon M \otimes^{\mathbb{L}}_A N \to M' \otimes^{\mathbb{L}}_{A'} N'$$
is a quasi-isomorphism if φ, f and g are.
2. (Flatness) The natural map $\epsilon\colon M \otimes^{\mathbb{L}}_A N \to M \otimes_A N$ is a quasi-isomorphism if M or N is flat as an A-module.

Corollary 8.50. *Under the hypotheses of Theorem 8.49, if $\Bbbk$ is a field of characteristic zero, then the induced morphism of cochain complexes*

$$\Omega^*(X) \otimes^{\mathbb{L}}_{\Omega^*(B)} \Omega^*(E) \to \Omega^*(X \times_B E)$$

is a quasi-isomorphism.

Proof. This is a consequence of the existence of a natural zig-zag of quasi-isomorphisms of cochain algebras $C^*(-;\Bbbk) \sim \Omega^*(-)$ (Theorem 7.14) and the homotopy invariance property of the derived tensor product. $\square$

Proposition 8.51. *Let (X, A) be a relative Sullivan algebra. Then X is flat as a left A-module.*

Theorem 8.52. *Under the hypotheses of Theorem 8.49, if $\Bbbk$ is a field of characteristic zero, then the diagram*

$$\begin{tikzcd}
\Omega^*(B) \arrow[r] \arrow[d] & \Omega^*(X) \arrow[d] \\
\Omega^*(E) \arrow[r] & \Omega^*(X \times_B E)
\end{tikzcd}$$

is a homotopy pushout in the category of commutative cochain algebras.

Proof. Let $\Omega^*(B) \overset{i}{\rightarrowtail} S \xrightarrow{\sim} \Omega^*(X)$ be a Sullivan model for the morphism $\Omega^*(f)$. We need to show that the induced morphism of cochain algebras

$$q\colon \Omega^*(E) \otimes_{\Omega^*(B)} S \to \Omega^*(E \times_B X)$$

is a quasi-isomorphism. This follows from contemplating the following diagram:

$$\begin{tikzcd}
\Omega^*(E) \otimes^{\mathbb{L}}_{\Omega^*(B)} S \arrow[r, "\sim"] \arrow[d, "\sim"] & \Omega^*(E) \otimes^{\mathbb{L}}_{\Omega^*(B)} \Omega^*(X) \arrow[d, "\sim"] \\
\Omega^*(E) \otimes_{\Omega^*(B)} S \arrow[r, "\therefore\sim", "q"'] & \Omega^*(E \times_B X)
\end{tikzcd}$$

The top horizontal map is a quasi-isomorphism by the homotopy invariance property of the derived tensor product. The left vertical map is a quasi-isomorphism because S is flat as an $\Omega^*(B)$-module (Proposition 8.51). The right vertical map is a quasi-isomorphism by Corollary 8.50. $\square$

An important special case of Theorem 8.52 is when X is point. In this case, specifying a map $f\colon X \to B$ amounts to choosing a point $b \in B$, and the fiber product $X \times_B E$ is simply the fiber $p^{-1}(b)$.

Theorem 8.53 (Fibration theorem)**.** *Let $\Bbbk$ be a field of characteristic zero, and let $F \to E \to B$ be a fibration of path connected spaces where B is simply connected and E, B are of finite $\Bbbk$-type. Then*

$$\Omega^*(B) \to \Omega^*(E) \to \Omega^*(F)$$

is a homotopy cofiber sequence of commutative cochain algebras.

We will now give the proof of Theorem 8.46.

Proof of Theorem 8.46. Let $\mathscr{C}$ denote the class of commutative cochain algebras A for which $\eta_A\colon A \to \Omega^*\langle A\rangle$ is a quasi-isomorphism.

1. *The initial cochain algebra $\mathbb{Q}$ belongs to $\mathscr{C}$.* In fact, the map $\eta_{\mathbb{Q}}$ is an isomorphism because $\langle \mathbb{Q}\rangle = *$ and $\Omega^*(*) = \mathbb{Q}$.

2. *If $f\colon A \xrightarrow{\sim} B$ is a quasi-isomorphism between cofibrant cochain algebras, then A belongs to $\mathscr{C}$ if and only if B does.* Indeed, any quasi-isomorphism between cofibrant cochain algebras is a homotopy equivalence (this follows easily from Proposition 8.34). Since the spatial realization functor preserves homotopies (Lemma 8.42), it follows that the induced simplicial map $\langle f\rangle\colon \langle B\rangle \to \langle A\rangle$ is a homotopy equivalence. In particular, $\langle f\rangle$ induces an isomorphism in rational cohomology. Hence, the right vertical map below is a quasi-isomorphism.

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "\sim"', "f"] & \Omega^*\langle A\rangle \arrow[d, "\sim"', "\Omega^*\langle f\rangle"] \\
B \arrow[r, "\eta_B"] & \Omega^*\langle B\rangle
\end{tikzcd}$$

   This proves the claim.

3. *The disk $D(V)$ belongs to $\mathscr{C}$ for every graded vector space V.* The disk $D(V)$ is cofibrant and quasi-isomorphic to $\mathbb{Q}$, so this follows by combining the two previous statements.

4. *The sphere $S(n)$ belongs to $\mathscr{C}$ for every $n \geq 2$.* By Corollary 8.45, $\langle S(n)\rangle \simeq K(\mathbb{Q}, n)$. The map $K(\mathbb{Z}, n) \to K(\mathbb{Q}, n)$ is a $\mathbb{Q}$-localization and in particular a rational cohomology isomorphism. By combining this with Theorem 5.1 and Theorem 7.14, we conclude that the cohomology of the cochain algebra $\Omega^*\langle S(n)\rangle$ is a free graded commutative algebra on a generator of degree n. Thus, the cohomology algebras of $\Omega^*\langle S(n)\rangle$ and $S(n)$ are abstractly isomorphic. To verify that $\eta_{S(n)}\colon S(n) \to \Omega^*\langle S(n)\rangle$ induces an isomorphism in cohomology, it suffices to check that $\mathrm{H}^n(\eta_{S(n)})$ is non-trivial. (Indeed, any non-zero element of H^n generates the cohomology algebra, and a morphism between free algebras on one generator is an isomorphism if it maps a generator to a generator.) That $\mathrm{H}^n(\eta_{S(n)})$ is non-trivial follows from the fact that the homotopy class of $\eta_{S(n)}$ corresponds to the identity map of $\langle S(n)\rangle$ under the bijection
$$[S(n), \Omega^*\langle S(n)\rangle] \cong [\langle S(n)\rangle, \langle S(n)\rangle]$$
of Proposition 8.44.

5. *Consider a pushout diagram of connected commutative cochain algebras of finite type*
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & B \arrow[d] \\ X \arrow[r] & X \otimes_A B \end{tikzcd} \tag{17}$$
where (X, A) is a relative Sullivan algebra and A is simply connected. If X, A, B belong to $\mathscr{C}$, then so does $X \otimes_A B$.

To prove this, we first apply the spatial realization functor to get a pullback diagram of simplicial sets
$$\begin{tikzcd} \langle X \otimes_A B\rangle \arrow[r] \arrow[d] & \langle X\rangle \arrow[d, "\langle i\rangle"] \\ \langle B\rangle \arrow[r] & \langle A\rangle \end{tikzcd}$$
The vertical maps are (rational) Kan fibrations, since i is a cofibration (being the inclusion of a relative Sullivan algebra). Next, consider the following diagram.
$$\begin{tikzcd} X \otimes^{\mathbb{L}}_A B \arrow[r, "\sim", "\eta_X \otimes^{\mathbb{L}}_{\eta_A} \eta_B"'] \arrow[d, "\sim"] & \Omega^*\langle X\rangle \otimes^{\mathbb{L}}_{\Omega^*\langle A\rangle} \Omega^*\langle B\rangle \arrow[d, "\sim"] \\ X \otimes_A B \arrow[r, "\therefore\sim", "\eta_{X\otimes_A B}"'] & \Omega^*\langle X \otimes_A B\rangle \end{tikzcd}$$
The top horizontal map is a quasi-isomorphism by the homotopy invariance property of the derived tensor product, and the hypothesis that X,

A and B belong to $\mathscr{C}$. The right vertical map is a quasi-isomorphism by Corollary 8.50 (note that $\langle X \otimes_A B\rangle \cong \langle X\rangle \times_{\langle A\rangle} \langle B\rangle$). The left vertical map is a quasi-isomorphism because X is flat as an A-module (Proposition 8.51). It follows that $X \otimes_A B$ belongs to $\mathscr{C}$.

6. *The sphere $S(V)$ belongs to $\mathscr{C}$ for every graded vector space $V = V^{\geq 2}$ of finite type.* If $\{x_i\}_{i\in I}$ is a homogeneous basis for V, then $S(V) \cong \otimes_{i\in I} S(|x_i|)$. Thus, if V is finite dimensional, i.e., I is finite, then the claim follows by induction, using 4 and 5 above. If V is of finite type, then $V^{\leq N}$ is finite dimensional for each positive integer N, so by what we just said $S(V^{\leq N})$ belongs to $\mathscr{C}$ for every N. Now fix k and choose some integer N with $N > k$. The inclusion $i\colon S(V^{\leq N}) \to S(V)$ induces an isomorphism in cohomology in degrees $\leq N$, tautologically. Since $\langle S(V)\rangle \simeq \prod_{\ell\geq 2} K(V^\ell, \ell)$, we also have that the map $\langle i\rangle\colon \langle S(V)\rangle \to \langle S(V^{\leq N})\rangle$ induces an isomorphism on $\pi_{\leq N}$. It follows from the (absolute) Whitehead theorem (see Theorem 3.6) that $\langle i\rangle$ induces an isomorphism also on $\mathrm{H}^{<N}$. In particular, both morphisms i and $\Omega^*\langle i\rangle$ induce an isomorphism on H^k. Therefore, since $S(V^{\leq N})$ belongs to $\mathscr{C}$, it follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}^k\, S(V^{\leq N}) \arrow[r, "\cong", "\mathrm{H}^k(\eta_{S(V^{\leq N})})"'] \arrow[d, "\cong"', "\mathrm{H}^k(i)"] & \mathrm{H}^k\, \Omega^*\langle S(V^{\leq N})\rangle \arrow[d, "\mathrm{H}^k\,\Omega^*\langle i\rangle"', "\cong"] \\
\mathrm{H}^k\, S(V) \arrow[r, "\therefore\cong", "\mathrm{H}^k(\eta_{S(V)})"'] & \mathrm{H}^k\, \Omega^*\langle S(V)\rangle
\end{tikzcd}$$

that $\mathrm{H}^k(\eta_{S(V)})$ is an isomorphism. The argument just given can be repeated for every k, so it follows that $\eta_{S(V)}$ is a quasi-isomorphism.

7. *Every simply connected cofibrant commutative cochain algebra A with cohomology of finite type belongs to $\mathscr{C}$.* This requires some facts about minimal Sullivan models to be proved in the next section. Under the stated hypothesis, we can construct a minimal Sullivan model $X \xrightarrow{\sim} A$ such that $X^0 = \mathbb{Q}$, $X^1 = 0$ and X^p is finite dimensional for every p. By 2, it suffices to check that X belongs to $\mathscr{C}$. But $X = \cup_n X_n$ where $X_{-1} = \mathbb{Q}$ and X_n is obtained from X_{n-1} by a pushout as in Definition 8.3. Since X is of finite type, the graded vector spaces V_n are necessarily of finite type. Therefore, it follows by induction, using 1, 3, 6 and 5, that X_n belongs to $\mathscr{C}$ for every n. Since every component X^k is finite dimensional, it follows that $X^k = X_n^k$ for n large enough. In particular, for a given k, if we choose n large enough, then the inclusion $X_n \to X$ induces isomorphisms $\mathrm{H}^{\leq k}(X_n) \xrightarrow{\cong} \mathrm{H}^{\leq k}(X)$ and $\pi_{\leq k+1}\langle X\rangle \xrightarrow{\cong} \pi_{\leq k+1}\langle X_n\rangle$ (use Exercise 8.59 for the latter statement). Arguing as in 6, we get that $\mathrm{H}^k(\eta_X)$ is an isomorphism. Since this works for every k, it follows that η_X is a quasi-isomorphism, so that $X \in \mathscr{C}$.

$\square$

8.5 Minimal Sullivan models

The wordlength filtration

Let $\Bbbk$ be a field and let V be a graded vector space over $\Bbbk$. The free graded commutative algebra ΛV can be decomposed as

$$\Lambda V = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \cdots$$

where $\Lambda^k V = (V^{\otimes k})_{\Sigma_k}$ is the space of homogeneous graded commutative polynomials of degree k. Note that $\Lambda^0 V = \Bbbk$ and $\Lambda^1 V = V$.

The 'wordlength' filtration of ΛV is the decreasing filtration

$$\Lambda V \supset \Lambda^{\geq 1} V \supset \Lambda^{\geq 2} V \supset \cdots$$

where

$$\Lambda^{\geq k} V = \Lambda^k V \oplus \Lambda^{k+1} V \oplus \cdots .$$

Any map $f\colon \Lambda V \to \Lambda W$ that is filtration preserving in the sense that

$$f(\Lambda^{\geq k} V) \subseteq \Lambda^{\geq k} W$$

can be decomposed as

$$f = f_0 + f_1 + f_2 + \cdots$$

where f_r is homogeneous of degree r with respect to wordlength, i.e.,

$$f_r(\Lambda^k V) \subseteq \Lambda^{k+r} W$$

for all $k \geq 0$.

Clearly, for a composite of filtration preserving maps

$$\Lambda V \xrightarrow{f} \Lambda W \xrightarrow{g} \Lambda U$$

we have the relation

$$(gf)_n = \sum_{p+q=n} g_p f_q$$

for every $n \geq 0$.

Minimal cochain algebras

Consider a cochain algebra of the form $(\Lambda V, d)$, where V is a graded vector space with $V = V^{\geq 1}$, i.e., V is concentrated in positive cohomological degrees. Then the differential d is automatically filtration preserving. Indeed, since $\Lambda^0 V = \Bbbk$ is concentrated in cohomological degree 0, we must have $d(V) \subseteq \Lambda^{\geq 1} V$. Then it follows from the Leibniz rule that $d(\Lambda^k V) \subseteq \Lambda^{\geq k} V$ for all k. Thus, we can decompose d into wordlength homogeneous components

$$d = d_0 + d_1 + d_2 + \ldots, \quad d_r(\Lambda^k V) \subseteq \Lambda^{k+r} V.$$

The relation $d^2 = 0$ is equivalent to the infinite series of relations

$$\begin{aligned} d_0 d_0 &= 0, \\ d_0 d_1 + d_1 d_0 &= 0, \\ d_0 d_2 + d_1 d_1 + d_2 d_0 &= 0, \\ &\vdots \end{aligned}$$

In particular the restriction of d_0 to $\Lambda^1 V = V$ makes (V, d_0) into a cochain complex.

Definition 8.54. A cochain algebra of the form $(\Lambda V, d)$ with $V = V^{\geq 1}$ is called *minimal* if $d_0 = 0$. Equivalently, $(\Lambda V, d)$ is minimal if and only if $d(V) \subseteq \Lambda^{\geq 2} V$.

Let $f\colon (\Lambda V, d) \to (\Lambda W, \delta)$ be a morphism of cochain algebras with $V = V^{\geq 1}$ and $W = W^{\geq 1}$. Then f is automatically filtration preserving (why?), and we may decompose it as

$$f = f_0 + f_1 + \ldots, \quad f_r(\Lambda^k V) \subseteq \Lambda^{k+r} W.$$

That f is a morphism of cochain algebras means that

$$\mu(f \otimes f) = fm, \quad \delta f = fd \quad f(1) = 1, \tag{18}$$

where $m\colon \Lambda V \otimes \Lambda V \to \Lambda V$ and $\mu\colon \Lambda W \otimes \Lambda W \to \Lambda W$ are the multiplication maps. If we decompose the equations (18) into homogeneous parts, we get

$$\sum_{p+q=n} \mu(f_p \otimes f_q) = f_n m, \quad \sum_{p+q=n} \delta_p f_q = \sum_{p+q=n} f_p d_q, \quad f_n(1) = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases}$$

for every $n \geq 0$. In particular, for $n = 0$, we get that $f_0\colon (\Lambda V, d_0) \to (\Lambda W, \delta_0)$ is a morphism of cochain algebras.

Proposition 8.55. *Let $V = V^{\geq 2}$ and $W = W^{\geq 1}$. If*

$$f, g\colon (\Lambda V, d) \to (\Lambda W, \delta)$$

are homotopic morphisms of commutative cochain algebras, then the induced morphisms of cochain algebras

$$f_0, g_0\colon (\Lambda V, d_0) \to (\Lambda W, \delta_0)$$

are homotopic. In particular, the morphisms of cochain complexes $f_0, g_0\colon (V, d_0) \to (W, \delta_0)$ are chain homotopic.

Proof. Let $h\colon \Lambda V \to \Lambda W \otimes \Lambda(t, dt)$ be a cochain algebra homotopy between f and g, i.e., a morphism of cochain algebras such that $h|_{t=0} = g$ and $h|_{t=1} = f$. By the assumption that $V = V^{\geq 2}$, and the fact that $\Lambda(t, dt)^{\geq 2} = 0$, it follows that h is filtration preserving in the sense that

$$h(\Lambda^{\geq k} V) \subseteq \Lambda^{\geq k} W \otimes \Lambda(t, dt).$$

It follows that we may decompose h as $h = h_0 + h_1 + \cdots$ and it is easy to see that h_0 will be a cochain algebra homotopy between f_0 and g_0. It follows from Proposition 8.39 that $f_0, g_0\colon (\Lambda V, d_0) \to (\Lambda W, \delta_0)$ are homotopic as morphisms of cochain complexes. It is easy to see that the homotopy can be chosen to be homogeneous with respect to wordlength. In particular, restricting attention to wordlength 1, we get that $f_0, g_0\colon (V, d_0) \to (W, \delta_0)$ are homotopic as morphisms of cochain complexes. ☐

Lemma 8.56. *Let $f\colon \Lambda V \to \Lambda W$ be a filtration preserving morphism of graded algebras. Then f is an isomorphism if and only if $f_0\colon V \to W$ is an isomorphism.*

Proof. First we make an easy observation: We have the decomposition

$$f = f_0 + f_1 + f_2 + \cdots, \quad f_r(\Lambda^k V) \subseteq \Lambda^{k+r} W.$$

We observed before that $f_0\colon \Lambda V \to \Lambda W$ is a morphism of algebras. As such, it is induced by the linear map $f_0\colon V \to W$. Therefore $f_0\colon V \to W$ is an isomorphism if and only if the morphism of algebras $f_0\colon \Lambda V \to \Lambda W$ is an isomorphism. Having observed this, we can proceed to the proof.

$\Rightarrow$: If f is an isomorphism with inverse g, then because of the relation $(gf)_0 = g_0 f_0$, we have that f_0 is an isomorphism with inverse g_0.

$\Leftarrow$: Suppose that $f_0\colon \Lambda V \to \Lambda W$ is an isomorphism. We wish to find an inverse of f, i.e., a linear map $g\colon \Lambda W \to \Lambda V$ that solves the equations

$$fg = 1, \quad gf = 1.$$

If we decompose the first equation into homogeneous pieces

$$\begin{aligned} f_0 g_0 &= 1, \\ f_0 g_1 + f_1 g_0 &= 0, \\ f_0 g_2 + f_1 g_1 + f_2 g_0 &= 0, \\ &\vdots \end{aligned}$$

we can solve for g_r inductively. Let g_0 be the inverse of f_0. Then let

$$\begin{aligned} g_1 &= -g_0 f_1 g_0, \\ g_2 &= -g_0 f_1 g_1 - g_0 f_2 g_0, \\ &\vdots \end{aligned}$$

Putting $g = g_0 + g_1 + \cdots$, we get a solution to the equation $fg = 1$. As the reader may check, it will also be a solution to $gf = 1$. $\square$

Theorem 8.57. *If $V = V^{\geq 2}$ and $W = W^{\geq 2}$, then any quasi-isomorphism between minimal cochain algebras*

$$f\colon (\Lambda V, d) \to (\Lambda W, \delta)$$

is an isomorphism.

Proof. We will prove later that any minimal cochain algebra of the form $(\Lambda V, d)$ with $V = V^{\geq 2}$ is a Sullivan algebra. In particular both the source and target of f are cofibrant commutative cochain algebras. Any quasi-isomorphism between cofibrant cochain algebras is a homotopy equivalence, so f is a homotopy equivalence, and it follows from Proposition 8.55 that

$$f_0\colon (V, d_0) \to (W, \delta_0)$$

is a homotopy equivalence of cochain complexes. By the minimality hypothesis, $d_0 = 0$ and $\delta_0 = 0$, so it follows that f_0 must be an isomorphism. By Lemma 8.56, this implies that f itself must be an isomorphism. $\square$

Theorem 8.58. *Every commutative cochain algebra A with $\mathrm{H}^0(A) = \Bbbk$ and $\mathrm{H}^1(A) = 0$ admits a unique, up to isomorphism, minimal Sullivan model*

$$(\Lambda V, d) \xrightarrow{\sim} A$$

with $V = V^{\geq 2}$. Furthermore, if $\mathrm{H}^*(A)$ is of finite type, then V is of finite type.

Proof. The algorithm described in the proof of Theorem 8.7 produces a minimal Sullivan model. Given two minimal Sullivan models $(\Lambda V, d) \xrightarrow{\sim} A$ and $(\Lambda W, \delta) \xrightarrow{\sim} A$, it follows from Corollary 8.21 that $(\Lambda V, d)$ and $(\Lambda W, \delta)$ are homotopy equivalent. Then it follows from Theorem 8.57 that $(\Lambda V, d)$ and $(\Lambda W, \delta)$ are isomorphic. □

Exercise 8.59. Let $A = (\Lambda V, d)$ be a minimal Sullivan algebra with $V = V^{\geq 2}$ of finite type. Prove that there is a bijection

$$\pi_k(\langle A \rangle) \cong \mathrm{Hom}_{\Bbbk}(V^k, \Bbbk).$$

Corollary 8.60. *Let X be a simply connected simplicial set of finite $\mathbb{Q}$-type.*

1. *The space X admits a unique, up to isomorphism, minimal Sullivan model*

   $$M_X = (\Lambda V, d) \xrightarrow{\sim} \Omega^*(X),$$

   with $V = V^{\geq 2}$ a finite type graded $\mathbb{Q}$-vector space.

2. *The adjoint simplicial map $X \to \langle M_X \rangle$ is a $\mathbb{Q}$-localization.*

3. *The rational homotopy groups of X can be calculated as*

   $$\pi_k(X) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}}(V^k, \mathbb{Q}).$$

4. *Two simply connected spaces X and Y of finite $\mathbb{Q}$-type are rationally homotopy equivalent if and only if their minimal Sullivan models M_X and M_Y are isomorphic as cochain algebras.*

8.6 Sullivan models and Postnikov towers

Let X be a simply connected space of finite $\mathbb{Q}$-type, and let A be the minimal Sullivan model of X. Then A is of the form $A = (\Lambda V, d)$ where $V = V^{\geq 2}$ is a graded vector space of finite type. As we have seen, the rational homotopy groups of X can be calculated from the minimal model by the formula

$$\pi_k(X) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}}(V^k, \mathbb{Q}).$$

In fact, more is true: we can recover the whole rational Postnikov tower for X. Since $V = V^{\geq 2}$, we have that

$$d(V^n) \subseteq \Lambda(V^{<n})$$

for all n, for degree reasons. This implies that

$$A_n = (\Lambda(V^{\geq n}), d)$$

is a sub cochain algebra of A, and that there is a pushout diagram

$$\begin{tikzcd} S(V^n[n+1]) \arrow[r, "d"] \arrow[d] & A_{n-1} \arrow[d] \\ D(V^n[n+1]) \arrow[r] & A_n \end{tikzcd} \tag{19}$$

for every n. (Here $V^n[n+1]$ denotes the graded vector V^n concentrated in degree $n+1$.) Applying the spatial realization functor to the filtration

$$\mathbb{Q} = A_1 \subset A_2 \cdots \subset A_{n-1} \subset A_n \subset \cdots \subset \cup_n A_n = A,$$

we get a tower of fibrations

$$\langle A \rangle = \varprojlim \langle A_n \rangle \to \cdots \to \langle A_n \rangle \to \langle A_{n-1} \rangle \to \cdots \to \langle A_2 \rangle \to \langle A_1 \rangle = *. \tag{20}$$

The pushout (19) gives rise to a pullback

$$\begin{tikzcd} \langle A_n \rangle \arrow[r] \arrow[d] & \langle D(V^n[n+1]) \simeq * \arrow[d] \\ \langle A_{n-1} \rangle \arrow[r] & S(V^n[n+1]) \simeq K((V^n)^*, n+1) \end{tikzcd}$$

which exhibits $\langle A_n \rangle \to \langle A_{n-1} \rangle$ as a principal fibration with fiber $K((V^n)^*, n)$. Thus, the tower (20) is a Postnikov tower for $\langle A \rangle$. Since $X \to \langle A \rangle$ is a $\mathbb{Q}$-localization, the tower (20) may be viewed as a rational Postnikov tower for X.

9 Interlude: Model categories

Model categories were introduced by Quillen [21], and in fact Quillen's rational homotopy theory [20] was one of the first applications of model categories. For a good introduction to model categories, we refer the reader to [4] and [16]. We will recall the basic definitions here.

Definition 9.1. A *model category* is a category $\mathcal{C}$ together with three distinguished classes of maps,

- *weak equivalences* $\xrightarrow{\sim}$,
- *fibrations* $\twoheadrightarrow$,
- *cofibrations* $\rightarrowtail$,

subject to the following axioms:

1. Limits and colimits exist in $\mathscr{C}$.
2. Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, if two out of the maps f, g and gf are weak equivalences, then so is the third.
3. If f is a retract of g and g is a weak equivalence, fibration, or a cofibration, then so is f.

4. The lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

can be solved whenever i is a cofibration, p is a fibration, and at least one of i and p is a weak equivalence.

5. Every map $f\colon X \to Y$ can be factored in two ways:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "i"'] & & Y \\ & Z \arrow[ur, two heads, "p"', "\sim"] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "j"', "\sim"] & & Y \\ & W \arrow[ur, two heads, "q"'] & \end{tikzcd}$$

The *homotopy category* of a model category $\mathcal{C}$ is the category $\mathrm{Ho}(\mathcal{C})$ obtained from $\mathcal{C}$ by formally inverting all weak equivalences.

Much of what happened in the previous section can be summarized by the following theorem.

Theorem 9.2. *Let $\Bbbk$ be a field of characteristic zero. The category of commutative cochain algebras $\mathbf{CDGA}_{\Bbbk}$ admits the structure of a model category where*

- *the weak equivalences are the quasi-isomorphisms,*
- *the fibrations are the surjective morphisms,*
- *the cofibrations are the cofibrations of cochain algebras, as in Definition 8.10.*

Definition 9.3. A *Quillen adjunction* between two model categories $\mathcal{C}$ and $\mathcal{D}$ is an adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

such that the left adjoint F preserves cofibrations and the right adjoint G preserves fibrations.

Every Quillen adjunction induces an adjunction between the associated homotopy categories,

$$\mathrm{Ho}(\mathcal{C}) \underset{RG}{\overset{LF}{\rightleftarrows}} \mathrm{Ho}(\mathcal{D})$$

and the Quillen adjunction is called a *Quillen equivalence* if this induced adjunction is an equivalence of categories.

Sullivan did not use the language of model categories in [25]. Theorem 9.2 was first stated explicitly in the AMS memoir [1] by Bousfield and Gugenheim.

10 Differential graded Lie algebras

Quillen's model [20] for rational homotopy theory using differential graded Lie algebras predates Sullivan's [25], and it was one of the first applications of the theory of model categories. This section is based on [20]. See also [27] and [5, Part IV].

10.1 Differential graded Lie algebras

Let $\Bbbk$ be a field of characteristic zero.

Definition 10.1. A differential graded (dg) Lie algebra is a (potentially unbounded) chain complex of vector spaces over $\Bbbk$,

$$\cdots \to L_{n+1} \xrightarrow{d} L_n \xrightarrow{d} L_{n-1} \to \cdots ,$$

together with a binary operation

$$[-,-]\colon L_p \otimes L_q \to L_{p+q},$$

called the 'Lie bracket', subject to the following axioms:

- (Anti-symmetry) $[x,y] = -(-1)^{pq}[y,x]$,
- (Leibniz rule) $d[x,y] = [dx,y] + (-1)^p[x,dy]$,
- (Jacobi identity) $[x,[y,z]] = [[x,y],z] + (-1)^{pq}[y,[x,z]]$,

for all $x \in L_p$, $y \in L_q$ and $z \in L_r$.

A *chain Lie algebra* is a differential graded algebra L such that $L = L_{\geq 0}$, i.e., $L_n = 0$ for $n < 0$. A chain Lie algebra L is called *connected* if $L = L_{\geq 1}$.

Example 10.2.

1. Every chain complex L can be given the structure of a dg Lie algebra by letting the bracket be identically zero. Such dg Lie algebras are called *abelian*.

2. Every associative dg algebra A may be viewed as a dg Lie algebra A^{Lie} with the *commutator Lie bracket*, defined by

$$[a,b] = ab - (-1)^{pq}ba$$

for $a \in A_p$ and $b \in A_q$. In particular, A is a commutative dg algebra if and only if the dg Lie algebra A^{Lie} is abelian.

3. A derivation of degree p is a linear map $\theta\colon L_* \to L_{*+p}$ such that

$$\theta[y,z] = [\theta(y),z] + (-1)^{pq}[y,\theta(z)]$$

for all $y \in L_q$, $z \in L_r$. If θ and ρ are derivations of degree p and q, respectively, then their commutator

$$[\theta,\rho] = \theta \circ \rho - (-1)^{pq}\rho \circ \theta$$

is a derivation of degree $p+q$. The collection of all derivations form a dg Lie algebra $\operatorname{Der} L$, with differential $[d,-]$ and the commutator bracket.

The Jacobi identity can be formulated as saying that for every $x \in L_p$, the linear map

$$\mathrm{ad}_x = [x, -] \colon L_n \to L_{n+p}$$

is a derivation of degree p. In fact, the map

$$\mathrm{ad} \colon L \to \operatorname{Der} L, \quad x \mapsto \mathrm{ad}_x,$$

is a morphism of dg Lie algebras.

The free Lie algebra

The forgetful functor from the category of dg Lie algebras to the category of chain complexes admits a left adjoint $\mathbb{L}$.

$$\begin{tikzcd} \mathbf{Ch} \arrow[r, shift left, "\mathbb{L}"] & \mathbf{DGL} \arrow[l, shift left, "forget"] \end{tikzcd}$$

For a chain complex V, the *free Lie algebra* $\mathbb{L}(V)$ is characterized by the following universal property: There is a natural morphism of chain complexes $\eta_V \colon V \to \mathbb{L}(V)$ such that every morphism of chain complexes $V \to L$, into a dg Lie algebra L, extends uniquely to a morphism of dg Lie algebras $\mathbb{L}(V) \to L$.

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[dr] & \mathbb{L}(V) \arrow[d, dashed, "\exists!"] \\ & L \end{tikzcd}$$

The free Lie algebra $\mathbb{L}(V)$ may be constructed as follows. The tensor algebra (free associative algebra on V)

$$\mathbb{T}(V) = \bigoplus_{k \geq 0} V^{\otimes k}$$

becomes a dg Lie algebra with the commutator bracket. The free Lie algebra $\mathbb{L}(V)$ can be defined as the sub dg Lie algebra of $\mathbb{T}(V)^{Lie}$ generated by $V \subset \mathbb{T}(V)$. In particular, if we set $\mathbb{L}^k(V) = \mathbb{L}(V) \cap V^{\otimes k}$, we have a direct sum decomposition

$$\mathbb{L}(V) = \mathbb{L}^1(V) \oplus \mathbb{L}^2(V) \oplus \cdots .$$

The subspace $\mathbb{L}^k(V)$ consists of all elements of 'bracket length' k.

Products and coproducts

If L and M are dg Lie algebras, then the direct sum $L \oplus M$ becomes a dg Lie algebra with the coordinate-wise structure:

$$(L \oplus M)_n = L_n \oplus M_n$$

$$d(x, y) = (dx, dy)$$

$$[(x, y), (x', y')] = ([x, x'], [y, y']).$$

The direct sum $L \oplus M$ together with the projection maps $L \leftarrow L \oplus M \rightarrow M$ represent the product of L and M in the category of dg Lie algebras.

Coproducts, or *free products*, are harder to describe. The free product $L * M$ can be constructed as the quotient

$$L * M = \mathbb{L}(L \oplus M)/I$$

where I is the ideal generated by elements of the form

$$[x, y]_{\mathbb{L}} - [x, y]_L, \quad [z, w]_{\mathbb{L}} - [z, w]_M$$

for elements $x, y \in L$ and $z, w \in M$, where $[-,-]_{\mathbb{L}}$ denotes the Lie bracket in the free Lie algebra $\mathbb{L}(L \oplus M)$ and $[-,-]_L$ and $[-,-]_M$ denote the Lie brackets of L and M, respectively.

10.2 The universal enveloping algebra and dg Hopf algebras

The universal enveloping algebra

The functor that sends an associative dg algebra A to the dg Lie algebra A^{Lie} (i.e. A viewed as a dg Lie algebra with the commutator bracket) admits a left adjoint U.

$$\mathbf{DGL} \underset{(-)^{Lie}}{\overset{U}{\rightleftarrows}} \mathbf{DGA}$$

The *universal enveloping algebra* UL is the initial associative dg algebra that receives a morphism from L. In other words, there is a natural morphism of dg Lie algebras $\eta_L \colon L \to (UL)^{Lie}$, such that for every dg algebra A and morphism of dg Lie algebras $L \to A^{Lie}$, there is a unique morphism of dg algebras $UL \to A$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "\eta_L"] \arrow[dr] & UL \arrow[d, dashed, "\exists!"] \\ & A \end{tikzcd}$$

commutes. The universal property determines UL up to canonical isomorphism.

Exercise 10.3. 1. Prove that there is a natural isomorphism $U\mathbb{L}(V) \cong \mathbb{T}(V)$.

2. Prove that if L is an abelian Lie algebra, then $UL \cong \Lambda L$, the free graded commutative algebra.

3. Prove that there is a natural isomorphism of dg algebras

$$U(L \oplus M) \cong U(L) \otimes U(M)$$

for any dg Lie algebras L and M.

4. Prove that the universal enveloping algebra admits the presentation

$$UL \cong \mathbb{T}(L)/I,$$

where $I \subset \mathbb{T}(L)$ is the two-sided dg ideal generated by all elements of the form

$$x \otimes y - (-1)^{pq} y \otimes x - [x, y]$$

for $x \in L_p$ and $y \in L_q$.

What kind of dg algebras can occur as universal enveloping algebras of dg Lie algebras? To address this question, we note that UL has additional structure. The diagonal $\Delta\colon L \to L \oplus L$, sending x to (x, x), and the zero map $z\colon L \to 0$ are morphisms of dg Lie algebras. They make (L, Δ, z) into a cocommutative coalgebra in the category of dg Lie algebras. Since the universal enveloping algebra functor takes sums to tensor products, we automatically get that $(UL, U\Delta, Uz)$ is a cocommutative *dg Hopf algebra*.

Definition 10.4. A dg Hopf algebra is a chain complex H together with morphisms of chain complexes

- Product $\mu\colon H \otimes H \to H$,
- Unit $\eta\colon \Bbbk \to H$,
- Coproduct $\Delta\colon H \to H \otimes H$,
- Counit $H \to \Bbbk$,

such that (H, μ, η) is a dg associative algebra, (H, Δ, η) is a dg coassociative coalgebra, and $\Delta\colon H \to H \otimes H$ and $\eta\colon H \to \Bbbk$ are morphisms of dg algebras.

The dg Hopf algebra H is called *cocommutative* if $T\Delta = \Delta$, where $T\colon H \otimes H \to H \otimes H$ is the isomorphism $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$. Similarly, H is called *commutative* if $\mu T = \mu$.

Let H be a dg Hopf algebra. The *primitives* of H is the subspace

$$\mathcal{P}(H) = \{x \in H \mid \Delta(x) = x \otimes 1 + 1 \otimes x\} \subset H.$$

Exercise 10.5. Show that $\mathcal{P}(H) \subset H$ is closed under the commutator Lie bracket and the differential.

Thus, $\mathcal{P}\colon \mathbf{DGH} \to \mathbf{DGL}$ is a functor from dg Hopf algebras to dg Lie algebras.

Theorem 10.6 (Milnor-Moore)**.** *The functors U and $\mathcal{P}$ give an equivalence of categories*

$$\mathbf{DGL}_0 \underset{P}{\overset{U}{\rightleftarrows}} \mathbf{DGH}_{0,c}$$

between the categories of connected dg Lie algebras and connected cocommutative dg Hopf algebras.

Remark 10.7. There is an extension of the Milnor-Moore theorem to fields of positive characteristic. In characteristic 2 or 3, the definition of a dg Lie algebra has to be adjusted slightly, see [23].

The free graded commutative algebra functor Λ enjoys the property

$$\Lambda(V \oplus W) \cong \Lambda(V) \oplus \Lambda(W)$$

for chain complexes V and W. The diagonal $\Delta\colon V \to V \oplus V$ and the zero map $V \to 0$ make any chain complex V into a cocommutative coalgebra. It follows that ΛV is a cocommutative coalgebra in the category of dg commutative algebras, i.e., a commutative and cocommutative Hopf algebra.

Theorem 10.8 (Poncaré-Birkhoff-Witt)**.** *Let L be dg Lie algebra. The map*

$$\gamma\colon \Lambda L \to UL$$

given by

$$\gamma(x_1 \ldots x_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \pm x_{\sigma 1} \ldots x_{\sigma k}$$

is a natural isomorphism of dg coalgebras.

We warn the reader that the Poincaré-Birkhoff-Witt isomorphism $\Lambda L \to UL$ is not an isomorphism of algebras, except in the trivial case when L is abelian.

Hopf algebras from topology

Let X be a simply connected space. Composition of loops gives the based loop space ΩX the structure of a homotopy associative monoid. The monoid structure is compatible with the diagonal map $\Omega X \to \Omega X \times \Omega X$, so it follows that $\mathrm{H}_*(\Omega X;\mathbb{Q})$ is a connected cocommutative graded Hopf algebra. The product on $\mathrm{H}_*(\Omega X;\mathbb{Q})$ is called the *Pontryagin product*. By the Milnor-Moore theorem, it is isomorphic to the universal enveloping algebra of the Lie algebra of primitives. The following theorem describes the primitives.

Theorem 10.9 (Cartan-Serre)**.** *The Hurewicz homomorphism $\pi_*(\Omega X) \otimes \mathbb{Q} \to \mathrm{H}_*(\Omega X;\mathbb{Q})$ induces an isomorphism of graded Lie algebras*

$$\pi_*(\Omega X) \otimes \mathbb{Q} \xrightarrow{\cong} \mathcal{P}\,\mathrm{H}_*(\Omega X;\mathbb{Q}).$$

By the Milnor-Moore theorem, it follows that the graded Lie algebra $\pi_*(\Omega X)\otimes \mathbb{Q}$ determines and is determined by the graded cocommutative Hopf algebra $\mathrm{H}_*(\Omega X;\mathbb{Q})$.

Indecomposables and primitives

For a chain complex V, let

$$V^\vee = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$$

denote the dual. There are natural chain maps

$$V \to (V^\vee)^\vee, \quad V^\vee \otimes W^\vee \to (V \otimes W)^\vee.$$

In general, they are not isomorphisms, but they are if V and W are of finite type.

If C is a dg coalgebra with comultiplication $\Delta\colon C \to C \otimes C$ and counit $\epsilon\colon C \to \mathbb{k}$, then $C^\vee$ becomes a dg algebra with multiplication

$$\mu\colon C^\vee \otimes C^\vee \to (C \otimes C)^\vee \xrightarrow{\Delta^\vee} C^\vee$$

and unit $\eta\colon \mathbb{k} \cong \mathbb{k}^\vee \xrightarrow{\epsilon^\vee} C^\vee$.

If A is a dg algebra of finite type, with multiplication $\mu\colon A \otimes A \to A$, then $A^\vee$ becomes a dg coalgebra with comultiplication

$$\Delta\colon A^\vee \xrightarrow{\mu^\vee} (A \otimes A)^\vee \xleftarrow{\cong} A^\vee \otimes A^\vee$$

and counit $\epsilon\colon A^\vee \xrightarrow{\eta^\vee} \mathbb{k}^\vee \cong \mathbb{k}$.

Let $\epsilon\colon A \to \mathbb{k}$ be an augmented dg algebra with augmentation ideal $\overline{A} = \ker(\epsilon)$. There is a splitting $A \cong \mathbb{k} \oplus \overline{A}$. The *indecomposables* of A is the quotient chain complex

$$Q(A) = \overline{A}/\overline{A} \cdot \overline{A}.$$

Thus, there is an exact sequence

$$\overline{A} \otimes \overline{A} \xrightarrow{\mu} \overline{A} \to Q(A) \to 0.$$

For example, if A is a cochain algebra of the form $A = (\Lambda V, d)$, where V is a graded vector space concentrated in positive cohomological degrees, then there is a unique augmentation $\epsilon\colon A \to \mathbb{k}$ (sending V to zero) and

$$Q(A) \cong (V, d_0).$$

Dually, if $\eta\colon \mathbb{k} \to C$ is a coaugmented dg coalgebra with $\overline{C} = \operatorname{coker}(\eta)$, then $C = \mathbb{k} \oplus \overline{C}$, and we have the *reduced comultiplication*

$$\overline{\Delta}\colon \overline{C} \to \overline{C} \otimes \overline{C}$$

defined by $\overline{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$. The *primitives* of C is the chain complex

$$\mathcal{P}(C) = \ker(\overline{\Delta}) = \{x \in C \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

Thus, there is a short exact sequence

$$0 \to \mathcal{P}(C) \to \overline{C} \xrightarrow{\overline{\Delta}} \overline{C} \otimes \overline{C}$$

The dual $C^\vee$ is an augmented dg algebra, and we have the relation

$$Q(C^\vee) \cong \mathcal{P}(C)^\vee.$$

The Hurewicz homomorphism

In the following exercise, we will prove the Cartan-Serre theorem in the case when X is of finite $\mathbb{Q}$-type.

Exercise 10.10. Let X be a simply connected space of finite $\mathbb{Q}$-type and let $(\Lambda V, d)$ be its minimal model.

1. Prove that the map
$$\mathrm{H}^k(\Lambda V, d) \to V^k,$$
induced by the projection onto indecomposables $(\Lambda V, d) \to (V, d_0)$, is dual to the rational Hurewicz homomorphism
$$\pi_k(X) \otimes \mathbb{Q} \to \mathrm{H}_k(X; \mathbb{Q}).$$

2. Prove that there is a relative Sullivan model for the augmentation map $\epsilon\colon (\Lambda V, d) \to \mathbb{Q}$ of the form
$$\begin{tikzcd} (\Lambda V, d) \arrow[rr] \arrow[dr, "i"'] & & \mathbb{Q} \\ & (\Lambda V \otimes \Lambda \overline{V}, D) \arrow[ur, "\sim"'] & \end{tikzcd}$$
where $\overline{V}^k = V^{k+1}$, and such that for every $v \in V$,
$$D(v \otimes 1) = d(v) \otimes 1, \quad D(1 \otimes \overline{v}) = v \otimes 1 + z$$
for some $z \in \Lambda^{\geq 1} V \otimes \Lambda \overline{V}$. It follows that the cofiber of i is isomorphic to $(\Lambda \overline{V}, 0)$, *with trivial differential.* Use the fibration theorem to conclude that the based loop space ΩX has minimal model $(\Lambda \overline{V}, 0)$.

3. Combine the two previous exercises to show that the dual of the rational Hurewicz homomorphism
$$\mathrm{H}^*(\Omega X; \mathbb{Q}) \to (\pi_*(\Omega X) \otimes \mathbb{Q})^\vee$$
is equivalent to the projection onto indecomposables
$$\mathrm{H}^*(\Omega X; \mathbb{Q}) \twoheadrightarrow Q\, \mathrm{H}^*(\Omega X; \mathbb{Q}).$$
Dualize to conclude that the rational Hurewicz homomorphism
$$\pi_*(\Omega X) \otimes \mathbb{Q} \to \mathrm{H}_*(\Omega X; \mathbb{Q})$$
is injective with image the primitives $\mathcal{P}\, \mathrm{H}_*(\Omega X; \mathbb{Q})$.

4. Prove that the Hurewicz homomorphism
$$\pi_*(\Omega X) \otimes \mathbb{Q} \to \mathrm{H}_*(\Omega X; \mathbb{Q})$$
is a morphism of Lie algebras, where the left hand side has the Samelson Lie bracket, and the right hand side has the commutator Lie bracket. Finish the proof of the Cartan-Serre theorem.

Example 10.11. Consider the sphere S^n where $n \geq 2$. The graded Lie algebra $\pi_*(\Omega S^n) \otimes \mathbb{Q}$ is a free graded Lie algebra on a generator α of degree $n-1$;
$$\pi_*(\Omega S^n) \otimes \mathbb{Q} \cong \mathbb{L}(\alpha) = \begin{cases} \langle \alpha \rangle, & n \text{ odd} \\ \langle \alpha, [\alpha, \alpha] \rangle, & n \text{ even.} \end{cases}$$

It follows that there is an isomorphism of graded Hopf algebras

$$\mathrm{H}_*(\Omega S^n; \mathbb{Q}) \cong U\mathbb{L}(\alpha) \cong \mathbb{T}(\alpha).$$

For n even, this example illustrates that the Poincaré-Birkhoff-Witt isomorphism

$$\gamma\colon \Lambda\mathbb{L}(\alpha) \to U\mathbb{L}(\alpha)$$

is not a morphism of algebras. The left hand side is a free graded commutative algebra on two generators: $x = \alpha$ and $y = [\alpha, \alpha]$ of degrees $n-1$ and $2n-2$. The relation $x^2 = 0$ holds by graded commutativity as x has odd degree. But $\gamma(x) = \alpha$ and α^2 is non-zero in the tensor algebra $\mathbb{T}(\alpha)$.

Example 10.12. Let H be the cochain Hopf algebra

$$(T(x) \otimes \Lambda(y), dx = 0, dy = x^2), \quad |x| = 3, |y| = 5.$$

The comultiplication is determined by declaring that x and y are primitive. The dg Hopf algebra H is cocommutative but non-commutative: if it was graded commutative, then we would have $x^2 = 0$ as x is of odd degree.

The primitives of H is the dg Lie algebra

$$\mathcal{P}(H) = \langle x, y, x^2 \rangle,$$

where $dx = 0$, $dy = x^2$, $d(x^2) = 0$ and where the only non-trivial Lie bracket is given by $[x, x] = 2x^2$. It follows from the Milnor-Moore theorem that H is isomorphic to the universal enveloping algebra of $\mathcal{P}(H)$.

The indecomposables of H is the chain complex

$$Q(H) = \langle x, y \rangle$$

with zero differential. The cohomology of H is the commutative and cocommutative Hopf algebra

$$\mathrm{H}_*(H) = \Lambda x.$$

In particular, $\mathrm{H}^*(QH) \ncong Q\,\mathrm{H}^*(H)$. It follows that the dual $H^\vee$ is an example of a commutative, non-cocommutative, chain Hopf algebra, whose homology is cocommutative, but for which

$$\mathrm{H}_*(\mathcal{P}H^\vee) \ncong \mathcal{P}\,\mathrm{H}_*(H^\vee).$$

10.3 Quillen's dg Lie algebra

Combining the Cartan-Serre theorem and the Milnor-Moore theorem, we get an isomorphism of graded Hopf algebras

$$\mathrm{H}_*(\Omega X; \mathbb{Q}) \cong U\pi_*(\Omega X) \otimes \mathbb{Q}. \tag{21}$$

One might get the idea that there should be chain level version of this statement. There should exist a dg Lie algebra $\lambda(X)$, depending functorially on X, with the following properties.

1. There is a natural isomorphism of graded Lie algebras

$$\mathrm{H}_*(\lambda(X)) \cong \pi_*(\Omega X) \otimes \mathbb{Q}.$$

2. There is a natural equivalence of dg Hopf algebras

$$C_*(\Omega X;\mathbb{Q}) \sim U\lambda(X)$$

that induces the isomorphism (21) in homology.

One might be tempted to try $\lambda(X) = \mathcal{P}C_*(\Omega X;\mathbb{Q})$. After all, the normalized chains $C_*(\Omega X;\mathbb{Q})$ is a dg Hopf algebra (provided one chooses a strictly associative model for the based loop space), so the space of primitives $\mathcal{P}C_*(\Omega X;\mathbb{Q})$ is a dg Lie algebra. The problem is that $C_*(\Omega X;\mathbb{Q})$ is not cocommutative, it is only cocommutative up to homotopy (in fact it is an E_∞-coalgebra). Therefore, we have no guarantee that $\mathrm{H}_*(\mathcal{P}C_*(\Omega X;\mathbb{Q}))$ is isomorphic to $\mathcal{P}\,\mathrm{H}_*(\Omega X;\mathbb{Q})$ (cf. Example 10.12).

Quillen's solution to this problem is to work simplicially until the very end. For a simplicial set X with trivial 1-skeleton, he defines

$$\lambda(X) = N_*\mathcal{P}\mathbb{Q}[GX]^\wedge_I.$$

Here GX denotes the Kan loop group of X; this is a simplicial group that models the based loop space ΩX. Next, $\mathbb{Q}[GX]^\wedge_I$ denotes the group algebra completed at the augmentation ideal — this is a simplicial complete Hopf algebra — and $\mathcal{P}\mathbb{Q}[GX]^\wedge_I$ denotes the simplicial Lie algebra of primitive elements. Finally, N_* is the normalized chains functor from simplicial vector spaces to chain complexes. This functor takes simplicial Lie algebras to dg Lie algebras.

10.4 Twisting morphisms

Let C be a cocommutative coaugmented dg coalgebra and let L be a dg Lie algebra. In this section, we will assume that C is simply connected in the sense that $C = \Bbbk \oplus C_{\geq 2}$, and we will assume that L is connected in the sense that $L = L_{\geq 1}$.

The chain complex $\mathrm{Hom}(C, L)$ with differential

$$\partial(f) = d_L \circ f - (-1)^{|f|} f \circ d_C$$

can be endowed with the structure of a dg Lie algebra with the *convolution Lie bracket*, defined as follows. For $f, g \in \mathrm{Hom}(C, L)$, the bracket $[f, g] \in \mathrm{Hom}(C, L)$ is the composite map

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} L \otimes L \xrightarrow{[-,-]} L$$

Exercise 10.13. Check that the above makes $(\mathrm{Hom}(C, L), \partial, [-,-])$ into a dg Lie algebra.

Definition 10.14. A *twisting morphism* $\tau\colon C \to L$ is a degree -1 map such that the 'Maurer-Cartan equation'

$$\partial(\tau) + \frac{1}{2}[\tau, \tau] = 0$$

is fulfilled in $\mathrm{Hom}(C, L)$. We will let $\mathrm{Tw}(C, L)$ denote the set of twisting morphisms from C to L.

For a fixed C, we get a contravariant functor $\mathrm{Tw}(-,C)$ from dg Lie algebras to sets, and for a fixed L, we have a covariant functor $\mathrm{Tw}(L,-)$ from the category of coaugmented cocommutative dg coalgebras to sets.

Proposition 10.15. *1. For a fixed dg Lie algebra L, the functor* $\mathrm{Tw}(-,L)$ *is representable; there is a cocommutative coaugmented dg coalgebra $\mathscr{C}(L)$ and a universal twisting morphism $\tau_L\colon \mathscr{C}(L)\to L$ with the property that for every twisting morphism $\tau\colon C\to L$, there is a unique morphism of coaugmented dg coalgebras $\psi_\tau\colon C\to\mathscr{C}(L)$ such that the diagram*

$$\begin{tikzcd} & \mathscr{C}(L) \arrow[dr, "\tau_L"] & \\ C \arrow[ur, dashed, "\psi_\tau"] \arrow[rr, "\tau"'] & & L \end{tikzcd}$$

is commutative.

2. *For a fixed dg coalgebra C, the functor* $\mathrm{Tw}(C,-)$ *is corepresentable; there is a dg Lie algebra $\mathscr{L}(C)$ and a universal twisting morphism $\tau_C\colon C\to\mathscr{L}(C)$ with the property that for every twisting morphism $\tau\colon C\to L$, there is a unique morphism of dg Lie algebras $\phi_\tau\colon\mathscr{L}(C)\to L$ such that the diagram*

$$\begin{tikzcd} C \arrow[rr, "\tau"] \arrow[dr, "\tau_C"'] & & L \\ & \mathscr{L}(C) \arrow[ur, dashed, "\phi_\tau"'] & \end{tikzcd}$$

is commutative.

In other words, the universal twisting morphisms give rise to natural bijections

$$\mathrm{Hom}_{dgl}(\mathscr{L}(C),L)\xrightarrow[\tau_C^*]{\cong}\mathrm{Tw}(C,L)\xleftarrow[(\tau_L)_*]{\cong}\mathrm{Hom}_{dgc}(C,\mathscr{C}(L))$$

Corollary 10.16. *There is an adjunction*

$$\mathbf{DGC}\underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}}\mathbf{DGL} \tag{22}$$

between the category of dg Lie algebras and the category of coaugmented connected cocommutative dg coalgebras.

Let

$$\mathscr{L}(C)=(\mathbb{L}(s^{-1}\overline{C}),\delta=\delta_0+\delta_1)$$

where the differential is the sum of two derivations δ_0 and δ_1 given by the formulas

$$\begin{aligned}\delta_0(s^{-1}x)&=-s^{-1}d_C(x),\\ \delta_1(s^{-1}x)&=-\frac{1}{2}\sum(-1)^{|sx_i'|}[s^{-1}x_i',s^{-1}x_i''],\end{aligned}$$

for $x \in \overline{C}$, if $\overline{\Delta}(x) = \sum x_i' \otimes x_i''$. The map $\tau_C \colon C \to \mathscr{L}(C)$ is defined by $\tau_C(1) = 0$ and $\tau_C(x) = s^{-1}x$ for $x \in \overline{C}$.

Let

$$\mathscr{C}(L) = (\Lambda sL, d = d_0 + d_1),$$

where

$$d_0(sx_1 \wedge \ldots \wedge sx_k) = -\sum_{i=1}^{k} (-1)^{n_i} sx_1 \wedge \ldots \wedge sd_L(x_i) \wedge \ldots \wedge sx_k$$

and

$$d_1(sx_1 \wedge \ldots \wedge sx_k) = \sum_{i<j} (-1)^{|sx_i|+n_{ij}} s[x_i, x_j] \wedge sx_1 \wedge \ldots \widehat{sx}_i \ldots \widehat{sx}_j \ldots \wedge sx_k$$

The signs in the above formulas are given by

$$n_i = \sum_{j<i} |sx_j|,$$

$$sx_1 \wedge \ldots \wedge sx_k = (-1)^{n_{ij}} sx_i \wedge sx_j \wedge sx_1 \wedge \ldots \widehat{sx}_i \ldots \widehat{sx}_j \ldots \wedge sx_k.$$

The map $\tau_L \colon \mathscr{C}(L) \to L$ is defined by $\tau_L(sx) = x$ and $\tau_L(sx_1 \wedge \ldots \wedge sx_k) = 0$ for $k \neq 1$.

Theorem 10.17. *1. The unit and counit morphisms*

$$C \to \mathscr{C}\mathscr{L}(C), \quad \mathscr{L}\mathscr{C}(L) \to L$$

are quasi-isomorphisms.

2. The functors $\mathscr{C}$ and $\mathscr{L}$ preserve quasi-isomorphisms.

There are model category structures on **DGL** and **DGC**, and Theorem 10.17 proves that the adjunction

Principal H-bundles and the proof of Theorem 10.17

There are different ways of proving Theorem 10.17. We will outline Quillen's original proof [20], which uses the notion of principal H-bundles for dg Hopf algebras H. In what follows, it will be useful to have the following dictionary to support the intuition:

Algebra	Topology
dg coalgebra C	topological space X
dg Hopf algebra H	topological group G

A right action of a dg Hopf algebra H on a dg coalgebra C is a morphism of dg coalgebras $C \otimes H \to H$, $c \otimes h \mapsto ch$ such that $(ch)h' = c(hh')$ and $c1 = c$ for all $c \in C$ and $h, h' \in H$.

Definition 10.18. Let H be a dg Hopf algebra. A *principal H-bundle* is a morphism of dg coalgebras

$$\pi \colon E \to C$$

together with a right action of H on E such that, after forgetting differentials, there is an isomorphism of graded coalgebras and H-modules $E \cong C \otimes H$.

Given a twisting morphism $\tau\colon C \to L$, there is an associated principal UL-bundle

$$C \otimes^\tau UL \to C$$

whose underlying graded coalgebra is $C \otimes UL$ with the evident right UL-module structure, but whose differential is

$$d^\tau = d_C \otimes 1 + 1 \otimes d_{UL} + \nabla_\tau,$$

where ∇_τ is defined as the composite

$$C \otimes UL \xrightarrow{\Delta_C \otimes 1} C \otimes C \otimes UL \xrightarrow{1 \otimes \iota\tau \otimes 1} C \otimes UL \otimes UL \xrightarrow{1 \otimes \mu_{UL}} C \otimes UL.$$

Here $\iota\tau$ is the composite of $\tau\colon C \to L$ and the inclusion $\iota\colon L \to UL$.

Exercise 10.19. Check that $(d^\tau)^2 = 0$ precisely because of the Maurer-Cartan equation for τ.

In particular, we have the universal bundles $\mathscr{C}(L) \otimes^{\tau_L} UL$ and $C \otimes^{\tau_C} U\mathscr{L}(C)$ constructed using the universal twisting morphisms. There are two basic assertions needed for the proof of Theorem 10.17:

1. The 'total spaces' of the universal bundles $\mathscr{C}(L) \otimes^{\tau_L} UL$ and $C \otimes^{\tau_C} U\mathscr{L}(C)$ are contractible, i.e., quasi-isomorphic to $\Bbbk$ via the counit morphism.
2. Given a morphism between principal bundles

$$\begin{tikzcd} H \arrow[r] \arrow[d, "f"] & E \arrow[r] \arrow[d, "g"] & C \arrow[d, "h"] \\ H' \arrow[r] & E' \arrow[r] & C', \end{tikzcd}$$

   if two out of f, g, h are quasi-isomorphisms, then so is the third.

These assertions should not come as a surprise, in view of the analogy with principal G-bundles, for G a topological group. The first assertion can be proved by writing down an explicit formula for a contracting homotopy. Given a principal H-bundle $H \to E \to C$, there is a spectral sequence of coalgebras

$$E^2_{p,q} = \mathrm{H}_p(C) \otimes \mathrm{H}_q(H) \Rightarrow \mathrm{H}_{p+q}(E),$$

and the second assertion follows from the comparison theorem for spectral sequences.

It is easy to derive Theorem 10.17 from the above assertions. Indeed, to prove that the counit morphism $\epsilon_L\colon \mathscr{L}\mathscr{C}(L) \to L$ is a quasi-isomorphism, consider the following morphism of principal bundles:

$$\begin{tikzcd} U\mathscr{L}\mathscr{C}(L) \arrow[r] \arrow[d, "U\epsilon_L"] & \mathscr{C}(L) \otimes^{\tau_{\mathscr{C}(L)}} U\mathscr{L}\mathscr{C}(L) \arrow[r] \arrow[d, "\sim"] & \mathscr{C}(L) \arrow[d, equal] \\ UL \arrow[r] & \mathscr{C}(L) \otimes^{\tau_L} UL \arrow[r] & \mathscr{C}(L). \end{tikzcd}$$

The middle vertical morphism is a quasi-isomorphism by the first assertion, and it follows from the second assertion that $U\epsilon_L$ is a quasi-isomorphism. The universal enveloping algebra functor reflects quasi-isomorphisms because $\mathrm{H}_*(U(L)) \cong U\,\mathrm{H}_*(L)$, so it follows that ϵ_L is a quasi-isomorphism. The remaining statements in Theorem 10.17 are proved in a similar manner.

Model category structure

The category **DGL** of connected chain Lie algebras admits a model category structure where

- The weak equivalences are the quasi-isomorphisms.
- The fibrations are the surjective morphisms.
- The cofibrations are those morphisms that have the left lifting property with respect to all fibrations.

The category **DGC** of simply connected cocommutative dg coalgebras admits a model structure where

- The weak equivalences are the quasi-isomorphisms.
- The cofibrations are the injective morphisms.
- The fibrations are the morphisms that have the right lifting property with respect to all cofibrations.

The adjunction

$$\mathbf{DGC} \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{DGL} \tag{23}$$

is a *Quillen adjunction* of model categories. This means that the left adjoint $\mathscr{L}$ preserves cofibrations and the right adjoint $\mathscr{C}$ preserves fibrations. Theorem 10.17 shows that the adjunction is a *Quillen equivalence.*

10.5 Cochain algebras vs. chain Lie algebras

If we restrict to finite type chain complexes, there is a contravariant equivalence

$$\mathbf{DGA}_f \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathbf{DGC}_f$$

between dg algebras of finite type and dg coalgebras of finite type. Composing with the adjunction between chain Lie algebras and dg coalgebras, we get a contravariant adjunction

$$\mathbf{CDGA}_f \underset{C^*}{\overset{L_*}{\rightleftarrows}} \mathbf{DGL}_f$$

between the categories of simply connected finite type commutative cochain algebras and finite type connected chain Lie algebras.

Proposition 10.20. *For every connected chain Lie algebra L of finite type, the commutative cochain algebra $C^*(L)$ is a simply connected Sullivan algebra.*

Proof. If V is a graded vector space of finite type, then there is an isomorphism $\Lambda(V)^\vee \cong \Lambda(V^\vee)$. It follows that $C^*(L)$ is isomorphic to a cochain algebra of the form $(\Lambda(V), d = d_0 + d_1)$, where $V = (sL)^\vee$ is concentrated in cohomological degrees ≥ 2. It is easy to check that any cochain algebra of this form is a Sullivan algebra. $\square$

Definition 10.21. A dg Lie algebra L is a *Lie model* for a simply connected space X if $C^*(L)$ is a Sullivan model for X, i.e., there is a quasi-isomorphism of cochain algebras $C^*(L) \xrightarrow{\sim} \Omega^*(X)$.

Example 10.22. For $n \geq 2$, the chain Lie algebra $(\mathbb{L}(\alpha), \delta\alpha = 0)$, where $|\alpha| = n - 1$, is a Lie model for the sphere S^n.

Let L be a dg Lie algebra and let $\alpha \in L_{n-1}$ be a cycle. Then we can attach a generator ξ of degree n to kill α. We let $(L[\xi], \delta\xi = \alpha)$ be the dg Lie algebra whose underlying graded Lie algebra is the free product $L * \mathbb{L}(\xi)$, and whose differential is determined by the requirement that $\delta\xi = \alpha$ and that the inclusion $L \to L * \mathbb{L}(\xi)$ is a morphism of dg Lie algebras.

Theorem 10.23. *Consider a cell attachment*

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ D^{n+1} \arrow[r] & X \cup_f D^{n+1} \end{tikzcd} .$$

If L is a Lie model for X, and if $\alpha \in L_{n-1}$ is a cycle representing the attaching map f, then

$$(L[\xi], \delta\xi = \alpha)$$

is a Lie model for the adjunction space $X \cup_f D^{n+1}$.

Proof. This follows from the fact that the cochain algebra functor $C^* \colon \mathbf{DGL} \to \mathbf{CDGA}$ takes homotopy pushouts to homotopy pullbacks, and the spatial realization functor $\langle - \rangle \colon \mathbf{CDGA} \to s\mathbf{Set}$ takes homotopy pullbacks to homotopy pushouts. $\square$

Using Theorem 10.23 iteratively, we can construct Lie models for any CW-complex.

Example 10.24. Consider the complex projective plane $\mathbb{CP}^2$. The standard cell decomposition has one cell each in dimensions $0, 2, 4$. The attaching map for the top cell is the Hopf map $\eta \colon S^3 \to S^2$; there is a pushout diagram

$$\begin{tikzcd} S^3 \arrow[r, "\eta"] \arrow[d] & S^2 \arrow[d] \\ D^4 \arrow[r] & \mathbb{CP}^2 \end{tikzcd} .$$

A Lie model for S^2 is given by $(\mathbb{L}(\iota), \delta\iota = 0)$, where ι has degree 1. We have the relation $2\eta = [\iota, \iota]$ in $\pi_3(S^2)$, where the right hand side denotes the Whitehead product of the identity map of S^2 with itself. Thus, the Hopf map is represented by the cycle $\frac{1}{2}[\iota, \iota] \in \mathbb{L}(\iota)$. It follows that

$$(\mathbb{L}(\iota, \xi), \delta\iota = 0, \delta\xi = \frac{1}{2}[\iota, \iota])$$

is a Lie mode for $\mathbb{CP}^2$.

Proposition 10.25. *If A is a finite type simply connected cochain algebra model for $\Omega^*(X)$, then $L_*(A)$ is a Lie model for X.*

Proof. It follows from Theorem 10.17 that the canonical morphism $C^*L_*(A) \to A$ is a quasi-isomorphism. Hence $C^*L_*(A)$ is a Sullivan model for A, and hence also for any commutative cochain algebra quasi-isomorphic to A. $\square$

Example 10.26. Consider the complex projective space $\mathbb{CP}^n$. This space is *formal* in the sense that $\Omega^*(\mathbb{CP}^n)$ is quasi-isomorphic to the cohomology $\mathrm{H}^*(\mathbb{CP}^n;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$, $|x| = 2$, viewed as a cochain algebra with zero differential. By Proposition 10.25, the chain Lie algebra $L_*(\mathbb{Q}[x]/(x^{n+1}))$ is a Lie model for $\mathbb{CP}^n$. By expanding the definition, we see that this dg Lie algebra admits the following explicit description:

$$(\mathbb{L}(\alpha_1, \alpha_2, \ldots, \alpha_n), \delta), \quad |\alpha_k| = 2k - 1,$$

where the differential is given by

$$\delta(\alpha_k) = \frac{1}{2} \sum_{p+q=k} [\alpha_p, \alpha_q].$$

For $n = 2$, this recovers the model for $\mathbb{CP}^2$ constructed above.

10.6 Loop spaces and suspensions

Rational homotopy theory belongs to the realm of unstable homotopy theory. A lot of information is lost when taking loops or suspensions.

Theorem 10.27. *The following are equivalent for a simply connected space X of finite $\mathbb{Q}$-type.*

1. *The space X is rationally homotopy equivalent to a product Eilenberg-MacLane space, i.e.,*

$$X \sim_{\mathbb{Q}} \prod_n K(\pi_n(X), n).$$

2. *The space X is rationally homotopy equivalent to a loop space, i.e.,*

$$X \sim_{\mathbb{Q}} \Omega Y,$$

for some 2-connected space Y.

3. *The rational Hurewicz homomorphism*

$$\pi_k(X) \otimes \mathbb{Q} \to \mathrm{H}_k(X;\mathbb{Q})$$

is injective for all $k > 0$.

4. *The cohomology algebra $\mathrm{H}^*(X;\mathbb{Q})$ is a free graded commutative algebra.*

Proof. $1 \Rightarrow 2$ is clear because a product of Eilenberg-MacLane spaces is a loop space.

$2 \Rightarrow 3$ follows from the Cartan-Serre theorem (Theorem 10.9).

$3 \Rightarrow 4$: By Exercise 10.10, the rational Hurewicz homomorphism is dual to the map in cohomology induced by the projection onto indecomposables $p\colon (\Lambda V, d) \to V$, where $(\Lambda V, d)$ is the minimal model of X. If the differential d in the minimal model is non-zero, then $d(v) \neq 0$ for some generator $v \in V^k$. But then v is not in the image of $p_*\colon \mathrm{H}^k(\Lambda V, d) \to V^k$ (why?), and then the Hurewicz homomorphism would not be injective. So if the Hurewicz homomorphism is injective, then $d = 0$ in the minimal model. But then $\mathrm{H}^*(X; \mathbb{Q}) \cong \mathrm{H}^*(\Lambda V, d) = \Lambda V$ is a free graded commutative algebra.

$4 \Rightarrow 1$: Suppose that $\mathrm{H}^*(X; \mathbb{Q})$ is free with algebra generators $\{x_i\}_{i \in I}$. Let V be the graded vector space spanned by all x_i, so that $\mathrm{H}^*(X; \mathbb{Q}) \cong \Lambda(V)$. Pick representative cocycles $\omega_i \in \Omega^*(X)$ with $[\omega_i] = x_i$, and define a morphism of cochain algebras

$$(\Lambda V, 0) \to \Omega^*(X)$$

by sending x_i to ω_i. This is a quasi-isomorphism because it induces the identity map in cohomology. Hence $S(V) = (\Lambda V, 0)$ is a minimal model for X. But then the adjoint map $X \to \langle S(V) \rangle$ is a rational homotopy equivalence, and $\langle S(V) \rangle$ is a product of Eilenberg-MacLane spaces. □

Theorem 10.28. *The following are equivalent for a simply connected space X of finite $\mathbb{Q}$-type.*

1. *The space X is rationally homotopy equivalent to a wedge of Moore spaces, i.e.,*
$$X \sim_{\mathbb{Q}} \bigvee_n M(\mathrm{H}_n(X), n).$$
2. *The space X is rationally homotopy equivalent to a suspension, i.e.,*
$$X \sim_{\mathbb{Q}} \Sigma Y,$$
for some connected space Y.
3. *The rational Hurewicz homomorphism*
$$\pi_k(X) \otimes \mathbb{Q} \to \mathrm{H}_k(X; \mathbb{Q})$$
is surjective for all $k > 0$.
4. *The homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ is a free graded Lie algebra.*

Proof. The proof is essentially dual to the proof of Theorem 10.27. □

Note that for a finitely generated abelian group, the Moore space $M(A, n)$ is rationally homotopy equivalent to $M(A \otimes \mathbb{Q}, n)$, which is rationally homotopy equivalent to a wedge of spheres $\vee^r S^n$, where $r = \operatorname{rank} A$. So the first condition in Theorem 10.28 can be reformulated as saying that X is rationally homotopy equivalent to a wedge of spheres.

11 Elliptic and hyperbolic spaces

This section is based on [5, §32-§33].

Let X be a simply connected space with finite dimensional rational cohomology algebra, i.e., the $\dim_{\mathbb{Q}} \mathrm{H}^*(X;\mathbb{Q}) < \infty$. The *formal dimension* of X is the largest integer N such that $\mathrm{H}^N(X;\mathbb{Q}) \neq 0$. For instance, if X is a simply connected closed manifold of dimension N, then X has formal dimension N.

Definition 11.1. The space X is called *elliptic* if $\dim_{\mathbb{Q}} \pi_*(X) \otimes \mathbb{Q} < \infty$, and it is called *hyperbolic* if $\dim_{\mathbb{Q}} \pi_*(X) \otimes \mathbb{Q} = \infty$.

Thus, every space X is either elliptic or hyperbolic.

Theorem 11.2. *Let X be an elliptic space of formal dimension N. Then*

1. *We have $\dim \pi_*(X) \otimes \mathbb{Q} \leq N$.*

2. *We have $\pi_k(X) \otimes \mathbb{Q} = 0$ for all $k > N$ except possibly $\pi_k(X) \otimes \mathbb{Q} \cong \mathbb{Q}$ for a single odd k in the range $N < k < 2N$.*

3. *There is an inequality $\dim \pi_{even}(X) \otimes \mathbb{Q} \leq \dim \pi_{odd}(X) \otimes \mathbb{Q}$, and the following are equivalent:*

   (a) *We have equality $\dim \pi_{even}(X) \otimes \mathbb{Q} = \dim \pi_{odd}(X) \otimes \mathbb{Q}$.*

   (b) *The Euler characteristic $\chi(X)$ is non-zero.*

   (c) *The rational cohomology $\mathrm{H}^*(X;\mathbb{Q})$ is concentrated in even degrees.*

   (d) *The rational cohomology algebra is an artinian complete intersection ring, i.e., there is an isomorphism*

   $$\mathrm{H}^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_p]/(f_1, \ldots, f_p)$$

   where $x_1, \ldots, x_p$ are generators of even degree, and $f_1, \ldots, f_p$ is a regular sequence in the polynomial algebra $\mathbb{Q}[x_1, \ldots, x_p]$.

Theorem 11.3. *Let X be a hyperbolic space of formal dimension N. Then the sequence of integers $(a_r)_{r \geq 1}$, where*

$$a_r = \dim \pi_{\leq r}(X) \otimes \mathbb{Q}$$

has exponential growth.

The proof of Theorem 11.3 relies on a deep analysis of the algebraic structure of the minimal Sullivan model of a hyperbolic space.

Exercise 11.4. Let X be a wedge of two spheres $S^n \vee S^m$, where $n, m \geq 2$. The homotopy Lie algebra of X is then a free graded Lie algebra,

$$\pi_*(\Omega X) \otimes \mathbb{Q} \cong \mathbb{L}(\alpha, \beta)$$

on two generators α and β of degrees $n-1$ and $m-1$.

1. Use the Poincaré-Birkhoff-Witt theorem to calculate $\dim_{\mathbb{Q}} \mathbb{L}(\alpha, \beta)_k$ for all k.

2. Use your calculation to verify that $\dim_{\mathbb{Q}} \pi_{\leq r}(X)$ has exponential growth.

11.1 Interlude: Regular sequences and Koszul complexes

We will recall some facts from commutative algebra about regular sequences and Koszul complexes that we will need for the proof of Theorem 11.2. The reader is referred to the first chapters of [2] for more details.

Let R be a commutative algebra over a field $\Bbbk$, and let $f \in R$ be an element. The element f is called *regular* if it is not a zero-divisor, i.e., if $rf = 0$ implies $r = 0$ for all $r \in R$.

The *Koszul complex* $K(f)$ is the following chain complex of R-modules

$$\begin{array}{ccc} 1 & & 0 \\ R & \xrightarrow{\cdot f} & R \end{array}$$

We make three trivial observations:

- There is an isomorphism of chain complexes of R-modules
$$K(f) \cong (R \otimes \Lambda(y), dy = f), \quad |y| = 1.$$
In particular, $K(f)$ has the structure of a dg algebra over R.
- The zeroth homology is the residue ring
$$\mathrm{H}_0(K(f)) = R/(f).$$
- The first homology is the annihilator of f in R,
$$\mathrm{H}_1(K(f)) = \{r \in R \mid rf = 0\}.$$
In particular, $\mathrm{H}_+(K(f)) = 0$ if and only if f is regular.

Next, consider a sequence of elements $f_1, \ldots, f_q \in R$. The sequence is called a *regular sequence* if for $i = 1, 2, \ldots, q$, the residue class

$$\overline{f}_i \in R/(f_1, \ldots, f_{i-1})$$

is a regular element in the residue ring.

The Koszul complex of a sequence $f_1, \ldots, f_q$ is defined to be the tensor product chain complex of R-modules

$$K(f_\bullet) = K(f_1, \ldots, f_q) = K(f_1) \otimes_R \ldots \otimes_R K(f_q).$$

We have the following extension of the above observations. The first two assertions are obvious, but the third assertion requires some work.

- There is an isomorphism of chain complexes of R-modules
$$K(f_\bullet) \cong (R \otimes \Lambda(y_1, \ldots, y_q), dy_1 = f_1, \ldots, dy_q = f_q).$$
In particular, $K(f_\bullet)$ has the structure of a dg algebra over R.
- The zeroth homology is the residue ring
$$\mathrm{H}_0(K(f_\bullet)) \cong R/(f_1, \ldots, f_q).$$

- $\mathrm{H}_+(K(f_\bullet)) = 0$ if and only if $f_1, \ldots, f_q$ is a regular sequence.

The third assertion follows from the following more general theorem (see [2, Theorem 1.6.17]).

The *grade* of an ideal $I \subseteq R$ is the length of a maximal regular sequence in I.

Theorem 11.5. *Suppose R is a noetherian ring. Let d denote the grade of the ideal $(f_1, \ldots, f_q) \subset R$, and let r denote the homological dimension of the Koszul complex, i.e., $\mathrm{H}_{>r}(K(f_\bullet)) = 0$ but $\mathrm{H}_r(K(f_\bullet)) \neq 0$. Then we have the relation*

$$d + r = q.$$

In particular, $\mathrm{H}_+(K(f_\bullet)) = 0$ if and only if $f_1, \ldots, f_q$ is a regular sequence.

Recall that the *Krull dimension* of a commutative noetherian $\Bbbk$-algebra S is the maximum length k of chains of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_k \subset S,$$

where the inclusions are proper. If I is an ideal in the polynomial ring $R = \Bbbk[x_1, \ldots, x_p]$ generated by polynomials $f_1, \ldots, f_q$, then the Krull dimension of the residue ring R/I is the same as the dimension of the algebraic variety $V(f_1, \ldots, f_q) \subset \mathbb{A}^p$ whose points are the common zeros of $f_1, \ldots, f_q$. The Krull dimension of the polynomial ring $\Bbbk[x_1, \ldots, x_p]$ is p.

Recall that a noetherian $\Bbbk$-algebra S is *artinian* if the following equivalent conditions are fulfilled:

1. S has Krull dimension 0.

2. S is finite dimensional as a vector space over $\Bbbk$.

The grade of an ideal in a polynomial ring can also be calculated in terms of Krull dimensions (see [2, Corollary 2.1.4]).

Theorem 11.6. *Let $R = \Bbbk[x_1, \ldots, x_p]$ and let $I \subset R$ be the ideal generated by the polynomials $f_1, \ldots, f_q \in R$. Let d denote the grade of the ideal I, and let k denote the Krull dimension of R/I. Then we have the relation*

$$d + k = p.$$

11.2 Elliptic spaces

In this section we will prove Theorem 11.2.

Fix a minimal Sullivan algebra

$$A = (\Lambda V, d),$$

where $V = V^{\geq 2}$ is finite dimensional. We do not yet assume that A has finite dimensional cohomology.

Definition 11.7. The *formal dimension* of A is the largest integer N such that $\mathrm{H}^N(A) \neq 0$. If no such integer exists, we say that A has formal dimension $N = \infty$.

It is possible to choose a basis for V,

$$v_1, \ldots, v_n$$

such that $d(v_i) \in \Lambda(v_1, \ldots, v_{i-1})$ for all i. In particular, $d(v_1) = 0$. Possibly after reordering, we can also write the basis as

$$x_1, \ldots, x_p, y_1, \ldots, y_q, \quad p + q = n,$$

where $x_1, \ldots, x_p$ have even degrees ≥ 2, and $y_1, \ldots, y_q$ have odd degrees ≥ 3. Thus,

$$\Lambda(V) \cong \mathbb{Q}[x_1, \ldots, x_p] \otimes \Lambda(y_1, \ldots, y_q).$$

We can decompose the differential d into homogeneous components with respect to wordlength in y,

$$d = d_{-1} + d_0 + d_1 + d_2 \ldots.$$

Here, d_r is the derivation characterized by

$$d_r(x_i) \in \mathbb{Q}[X] \otimes \Lambda^r(Y), \quad 1 \leq i \leq p$$

$$d_r(y_j) \in \mathbb{Q}[X] \otimes \Lambda^{1+r}(Y), \quad 1 \leq j \leq q.$$

In particular, d_{-1} decreases the y-wordlength by 1, so

$$d_{-1}(x_i) = 0, \quad d_{-1}(y_j) = f_j,$$

for certain polynomials $f_1, \ldots, f_q \in \mathbb{Q}[x_1, \ldots, x_p]$. It follows that $d_{-1}^2 = 0$, so

$$A^{pure} := (\Lambda(V), d_{-1})$$

is a Sullivan algebra.

Proposition 11.8. *The pure Sullivan algebra $A^{pure} = (\Lambda(V), d_{-1})$ is isomorphic to the Koszul complex*

$$(\Lambda(V), d_{-1}) \cong K(f_1, \ldots, f_q)$$

of the sequence of polynomials $f_1, \ldots, f_q \in \mathbb{Q}[x_1, \ldots, x_p]$.

Proposition 11.9. *Let $(\Lambda V, d)$ be a minimal Sullivan algebra with $V = V^{\geq 2}$ finite dimensional.*

1. *$(\Lambda V, d)$ and $(\Lambda V, d_{-1})$ have the same formal dimension N.*
2. *The formal dimension N is finite if and only if $\mathbb{Q}[x_1, \ldots, x_p]/(f_1, \ldots, f_q)$ is finite dimensional as a vector space over $\mathbb{Q}$. In this case,*
   - *$q \geq p$.*
   - *If we arrange so that $|x_1| \geq \ldots \geq |x_p|$ and $|f_1| \geq \ldots \geq |f_q|$, then*

     $$|f_k| \geq 2|x_k|$$

     for $k = 1, 2, \ldots, p$.

- *The formal dimension N can be calculated as*

$$N = \sum_{j=1}^{q} |f_j| - \sum_{i=1}^{p} |x_i| + p - q.$$

Granted this proposition, we can draw the following conclusions:

$$\begin{aligned}
N &= \sum_{j=1}^{q} |f_j| - \sum_{i=1}^{p} |x_i| + p - q && (|f_i| \geq 2|x_i|) \\
&\geq \sum_{j=1}^{q} |f_j| - \sum_{i=1}^{p} \frac{1}{2}|f_i| + p - q \\
&= \frac{1}{2}\sum_{j=1}^{q} |f_j| + \frac{1}{2}\sum_{i=p+1}^{q} |f_i| + p - q && (|f_j| = |y_j| + 1 \geq 4) \\
&\geq \frac{1}{2}\sum_{j=1}^{q} |f_j| 2(q-p) + p - q && (q \geq p) \\
&\geq \frac{1}{2}\sum_{j=1}^{q} |f_j|.
\end{aligned}$$

An immediate consequence of this inequality is that $N \geq \frac{1}{2}|f_j|$ for all j. In other words,

$$|y_j| \leq 2N - 1.$$

It also follows that at most one y_j can have degree $> N$. Furthermore, it follows that

$$|x_i| \leq \frac{1}{2}|f_i| \leq N$$

for all i. Finally, since $|f_j| \geq 4$ for all j, and $q \geq p$, we also get that

$$N \geq \frac{1}{2}\sum_{j=1}^{q} |f_j| \geq 2q \geq p + q.$$

If $(\Lambda V, d)$ is the minimal Sullivan model for an elliptic space X, then V is dual to the graded vector space $\pi_*(X) \otimes \mathbb{Q}$, and the above shows that

$$\dim \pi_{even}(X) \otimes \mathbb{Q} = p \leq q = \dim \pi_{odd}(X) \otimes \mathbb{Q},$$

$$\dim \pi_*(X) \otimes \mathbb{Q} = p + q \leq N,$$

and also that $\pi_{even}(X) \otimes \mathbb{Q}$ is concentrated in degrees $\leq N$, and $\pi_{odd}(X) \otimes \mathbb{Q}$ is concentrated in degrees $\leq 2N - 1$ with at most one basis element of degree $> N$. Thus, Proposition 11.9 implies the first two statements in Theorem 11.2.

Lemma 11.10. *Let $A = (\Lambda V, d)$ be a minimal Sullivan algebra of formal dimension $N < \infty$, where $V = V^{\geq 2}$ is finite dimensional. If $v \in V$ is a generator with $dv = 0$, then the formal dimension of the quotient A/v is finite and equals*

$$\begin{cases} N + |v| - 1, & |v| \text{ even}, \\ N - |v|, & |v| \text{ odd}. \end{cases}$$

Proof. <u>$|v|$ even</u>: There is a quasi-isomorphism $A[z|dz = v] \xrightarrow{\sim} A/v$, where we have added an exterior generator z of degree $|v| - 1$ to kill the cycle v. Consider the short exact sequence of chain complexes

$$0 \to A \to A[z|dz = v] \to A[z]/A \to 0.$$

Since $A[z] = A \oplus Az$ and $dz \in A$, the quotient chain complex $A[z]/A$ is isomorphic to A shifted by the degree of z. In particular, $\mathrm{H}^n(A[z]/A) \cong \mathrm{H}^{n-|v|+1}(A)$ for all n. Using this identification and $\mathrm{H}^n(A[z|dz = v]) \cong \mathrm{H}^n(A/v)$, a portion of the long exact cohomology sequence induced by the above short exact sequence looks like

$$\mathrm{H}^{n+|v|-1}(A) \to \mathrm{H}^{n+|v|-1}(A/v) \to \mathrm{H}^n(A) \xrightarrow{\partial} \mathrm{H}^{n+|v|}(A).$$

Since $|v| \geq 2$, the corner terms are zero for $n \geq N$, so $\mathrm{H}^{n+|v|-1}(A/v) \cong \mathrm{H}^n(A)$ for $n \geq N$. This proves the claim.

<u>$|v|$ odd</u>: Consider the short exact sequence

$$0 \to Av \to A \to A/v \to 0. \tag{24}$$

The reader should check that since $dv = 0$ and $v^2 = 0$, the ideal Av generated by v is, as a chain complex, isomorphic to A/v shifted by the degree of v. In particular, $\mathrm{H}^n(Av) \cong \mathrm{H}^{n-|v|}(A/v)$. The reader should also check that, under this identification, the connecting homomorphism $\partial\colon \mathrm{H}^n(A/v) \to \mathrm{H}^{n-|v|+1}(A/v)$ sends $[\overline{a}]$ to $[\overline{b}]$ if $a \in A$ is an element such that $da = bv$ in A. Here, $\overline{a}$ and $\overline{b}$ denote the residue classes of a and b in A/v.

Now we will prove that $\mathrm{H}^n(A) = 0$ for $n > N - |v|$. Under the identifications indicated above, a portion of the long exact sequence associated to (24) looks like

$$\mathrm{H}^n(A) \to \mathrm{H}^n(A/v) \xrightarrow{\partial} \mathrm{H}^{n-|v|+1}(A/v) \to \mathrm{H}^{n+1}(A).$$

This implies that the connecting homomorphism $\partial\colon \mathrm{H}^n(A/v) \to \mathrm{H}^{n-|v|+1}(A/v)$ is surjective for $n \geq N$ and an isomorphism for $n > N$. Thus, for every $n > N$ there is a sequence of isomorphisms, continuing indefinitely to the left,

$$\cdots \xrightarrow{\cong} \mathrm{H}^{n+|v|-1}(A/v) \xrightarrow{\cong} \mathrm{H}^n(A/v) \xrightarrow{\cong} \mathrm{H}^{n-|v|+1}(A/v).$$

We claim that the groups in this sequence are all zero. Indeed, if $w_0 \in A$ represents an element $[\overline{w}_0] \in \mathrm{H}^{n-|v|+1}$, then by chasing through the above sequence of isomorphisms, we can find an infinite sequence of elements $w_1, w_2, w_3, \ldots \in A$ such that

$$\cdots \quad dw_3 = w_2 v, \quad dw_2 = w_1 v, \quad dw_1 = w_0 v.$$

Since the differential d is minimal, we must have $\ell := \ell(w_0) \geq \ell(w_1) \geq \ell(w_2) \geq \cdots$, where $\ell(w_k)$ denotes the wordlength filtration degree of w_k[2]. At the same time we have $|w_0| < |w_1| < |w_2| < \cdots$. But since V is finite dimensional, there is an upper bound on the cohomological degree of elements $w \in \Lambda V$ with $\ell(w) \leq \ell$. This implies that $w_k = 0$ for k large enough, and hence $[\overline{w}_0] = \partial^k[\overline{w}_k] = 0$. We conclude that $\mathrm{H}^n(A) = 0$ for $n > N - |v|$.

[2] $\ell(w)$ is the greatest integer s for which $w \in \Lambda^{\geq s}(V)$

Finally, the corner terms are zero in the exact sequence

$$\mathrm{H}^{N-1}(A/v) \xrightarrow{\partial} \mathrm{H}^{N-|v|}(A/v) \to \mathrm{H}^{N}(A) \to \mathrm{H}^{N}(A/v),$$

which shows that $\mathrm{H}^{N-|v|}(A) \neq 0$. Thus, A/v has formal dimension $N - |v|$. $\square$

Proposition 11.11. *Let $(\Lambda V, d)$ be a minimal Sullivan algebra with $V = V^{\geq 2}$ finite dimensional. The following are equivalent:*

1. *$\mathrm{H}^*(\Lambda V, d)$ is finite dimensional.*
2. *$\mathrm{H}^*(\Lambda V, d_{-1})$ is finite dimensional.*
3. *$\mathbb{Q}[x_1, \ldots, x_p]/(f_1, \ldots, f_q)$ is finite dimensional.*

Proof. $\underline{2 \Rightarrow 3}$: We identify the pure Sullivan algebra $(\Lambda V, d_{-1})$ with the Koszul complex $K(f_\bullet)$. This has the form (let $R = \mathbb{Q}[x_1, \ldots, x_p]$)

$$\begin{array}{ccccccc} & & 2 & & 1 & & 0 \\ \cdots & \longrightarrow & \bigoplus_{i<j} Ry_i \wedge y_j & \xrightarrow{d_{-1}} & \bigoplus_i Ry_i & \xrightarrow{d_{-1}} & R \end{array}$$

If we momentarily forget about the cohomological grading of R, the Koszul complex is a non-negative chain complex of R-modules, where the homological degree is the same as the y-wordlength. The zeroth homology in this grading is

$$\mathrm{H}_0(K(f_\bullet)) = \mathbb{Q}[x_1, \ldots, x_p]/(f_1, \ldots, f_q) = R/I$$

where $I \subset R$ is the ideal generated by $f_1, \ldots, f_q$. Evidently, if $\mathrm{H}_*(K(f_\bullet))$ is finite dimensional, then so is R/I.

$\underline{3 \Rightarrow 2}$: This follows from the observation that the homology $\mathrm{H}_*(K(f_\bullet))$ is a finitely generated module over $\mathrm{H}_0(K(f_\bullet)) = R/I$. Indeed, $K(f_\bullet)$ is a finitely generated (free) R-module. Since R is noetherian, this implies that $\ker d_{-1} \subseteq K(f_\bullet)$ is finitely generated as an R-module. Since $I \subseteq \operatorname{im} d_{-1}$, this implies that $\ker d_{-1}/\operatorname{im} d_{-1}$ is a finitely generated R/I-module.

$\underline{1 \Rightarrow 3}$: We will prove by induction that for every even generator v_i, there is a integer $e_i \geq 2$ such that $v_i^{e_i} = d_{-1}(w)$, where w has y-wordlength 1. Then it will follow that R/I is finite dimensional. In the chosen order $v_1, \ldots, v_n$ of the basis for V, we have $dv_i \in \Lambda(v_1, \ldots, v_{i-1})$. In particular, $dv_1 = 0$. Assume that $|v_1|$ is even. By hypothesis, $\mathrm{H}^*(\Lambda V, d)$ is finite dimensional, so there is an integer $e_1 \geq 2$ and an element $\xi \in \Lambda(V)$ such that $v_1^{e_1} = d(\xi)$. If we decompose ξ into y-wordlength homogeneous components $\xi = \xi_1 + \xi_3 + \cdots$, then it follows that $v_1^{e_1} = d_{-1}(\xi_1)$. If $|v_1|$ is odd, then we do nothing and go to the next step.

In the next step, we mod out by v_1. By Lemma 11.10 the quotient, which is of the form $(\Lambda(v_2, \ldots, v_n), \overline{d})$, is a Sullivan algebra with finite formal dimension. If $|v_2|$ is even, then by iterating the above argument, we get that $v_2^{g_2} = d_{-1}(\omega)$ modulo v_1 for some integer g_2 and some ω of y-wordlength 1. If $|v_1|$ was odd, then it follows that $v_2^{g_2} = d_{-1}(\omega)$. If $|v_1|$ was even, then $v_2^{g_2} = d_{-1}(\omega) + v_1\zeta$, where ζ has y-wordlength 0. This implies that for r large, $(v_2^{g_2})^r$ is a d_{-1}-boundary of an element of y-wordlength 1. In the next step we mod out by v_2 and argue in a similar way to show that $v_3^{e_3} = d_{-1}(\eta)$ for some $e_3 \geq 2$ and some η of y-wordlength 1, if $|v_3|$ is even. Running through the sequence $v_1, \ldots, v_n$ in this way proves the claim.

2 $\Rightarrow$ 1: This follows immediately from the existence of a convergent IVth quadrant cohomological spectral sequence

$$E_1^{s,t} = \mathrm{H}_{-t}(K(f_\bullet))^{s+t} \Rightarrow \mathrm{H}^{s+t}(\Lambda V, d).$$

The spectral sequence is obtained from the decreasing filtration $\{F^s\}$ of $(\Lambda V, d)$, where

$$(F^s)^{s+t} = (\mathbb{Q}[x_1, \ldots, x_p] \otimes \Lambda^{\geq -t}(y_1, \ldots, y_q))^{s+t}.$$

The filtration is designed so that the associated graded only sees the pure part d_{-1} of the differential;

$$E_0^{*,*} \cong (\Lambda V, d_{-1}).$$

Convergence of the spectral sequence is ensured since V is finite dimensional. $\square$

Now we can finish the proof of Proposition 11.9.

$(\Lambda V, d)$ and $(\Lambda V, d_{-1})$ have the same formal dimension:

By Proposition 11.11, the formal dimensions of $A = (\Lambda V, d)$ and $A^{pure} = (\Lambda V, d_{-1})$ are either both infinite or both finite. If the formal dimension is finite, then we will prove that it is the same by induction on $\dim(V)$. It is obvious for $\dim(V) \leq 1$, because then $d = d_{-1} = 0$, so that $A = A^{pure}$. Let $n \geq 2$ and assume by induction that $\mathrm{fdim}(\Lambda W, \delta) = \mathrm{fdim}(\Lambda W, \delta_{-1})$ whenever $\dim(W) < n$. Recall that $A = (\Lambda V, d)$ is a Sullivan algebra with chosen basis $v_1, \ldots, v_n$ such that $dv_i \in \Lambda(v_1, \ldots, v_{i-1})$. In particular $dv_1 = 0$. Applying Lemma 11.10 twice with $v = v_1$, we get that

$$\mathrm{fdim}(A) = \mathrm{fdim}(A/v) + |v| = \mathrm{fdim}((A/v)^{pure}) + |v| = \mathrm{fdim}(A^{pure}/v) + |v| = \mathrm{fdim}(A^{pure}),$$

if $|v|$ is odd, where $\mathrm{fdim}(A/v) = \mathrm{fdim}((A/v)^{pure})$ by the induction hypothesis. And if $|v|$ is even we of course reach the same conclusion. Thus, we have proved $\mathrm{fdim}(A) = \mathrm{fdim}(A^{pure})$ as claimed.

Calculation of the formal dimension of $(\Lambda V, d)$:

We can use Lemma 11.10 on the sequence $x_1, \ldots, x_p \in A^{pure}$ to calculate the formal dimension of A:

$$\mathrm{fdim}(A) = \mathrm{fdim}(A^{pure}) = \mathrm{fdim}(A^{pure}/(x_1, \ldots, x_p)) - \sum_{i=1}^{p} (x_i - 1).$$

But $A^{pure}/(x_1, \ldots, x_p) \cong \Lambda(y_1, \ldots, y_q)$ with zero differential. The formal dimension of $\Lambda(y_1, \ldots, y_q)$ is the degree of the top dimensional element $y_1 \wedge \ldots \wedge y_k$. Thus,

$$\mathrm{fdim}(A^{pure}/(x_1, \ldots, x_p)) = \sum_{j=1}^{q} |y_j| = \sum_{j=1}^{q} |f_j| - q.$$

Putting the above facts together, we see that the formal dimension N of A may be calculated as

$$N = \sum_{j=1}^{q} |f_j| - \sum_{i=1}^{p} |x_i| + p - q$$

as claimed.

Arrange so that $|x_1| \geq \ldots \geq |x_p|$ and $|f_1| \geq \ldots \geq |f_q|$. We will then show that $|f_s| \geq 2|x_s|$ for all $1 \leq s \leq p$. Indeed, the algebra $\mathbb{Q}[x_1, \ldots, x_p]/(f_1, \ldots, f_q)$

is finite dimensional as a vector space. If we mod out by the indeterminates $x_{s+1}, \ldots, x_p$, we get a finite dimensional algebra of the form

$$\mathbb{Q}[x_1, \ldots, x_s]/(\overline{f}_1, \ldots, \overline{f}_q),$$

where $\overline{f}_j$ denotes the residue class of f_j modulo $(x_{s+1}, \ldots, x_p)$. This can be finite dimensional only if at least s out of the classes $\overline{f}_1, \ldots, \overline{f}_q$ are non-zero, or in other words, at least s out of $f_1, \ldots, f_q$ contain some monomial (necessarily quadratic or higher, because $dy_j = f_j$ and d is a minimal differential) involving only $x_1, \ldots, x_s$. Since $|x_1| \geq \ldots \geq |x_s|$, any monomial of that form must have cohomological degree at least $2|x_s|$. Thus, at least s out of $f_1, \ldots, f_q$ have cohomological degree at least $2|x_s|$. Since $|f_1| \geq \ldots \geq |f_q|$, this implies in particular that $|f_s| \geq 2|x_s|$.

11.3 Two open conjectures

If true, the following still open conjecture would give an explanation for the exponential growth of the dimensions of $\pi_*(X) \otimes \mathbb{Q}$ for hyperbolic spaces X.

Conjecture 11.12 (Avramov-Félix conjecture)**.** *If X is hyperbolic, then $\pi_*(\Omega X) \otimes \mathbb{Q}$ contains a free Lie algebra on two generators.*

While we are at it, let us mention another open conjecture concerning fibrations with elliptic fibers.

Conjecture 11.13 (Halperin conjecture)**.** *If X is elliptic with $\chi(X) \neq 0$, then the rational Serre spectral sequence of every fibration*

$$X \to E \to B,$$

with E, B simply connected, collapses at the E_2-page.

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