ISSN: 1401-5617

Cofinite Hochschild cohomology

Alexander Berglund

Research Reports in Mathematics
Number 1, 2007

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2007/1

Date of publication: January 8, 2007

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

COFINITE HOCHSCHILD COHOMOLOGY

ALEXANDER BERGLUND

Abstract. First steps are taken towards a cohomology theory of associative algebras R over a commutative noetherian ring k using 'cofinite cochains', or 'δ-cochains' as I will call them. These are defined using the subcomplex of the Hochschild cochain complex consisting of k-linear maps from $R^{\otimes n}$ to the coefficient module that factor through a quotient algebra of $R^{\otimes n}$ which is finitely generated as a k-module. Under certain reasonable conditions on R, it is possible to interpret cofinite cohomology, or 'δ-cohomology', as a derived functor. I show that if R is a commutative noetherian k-algebra fulfilling these conditions, then the natural map from δ-cohomology to Hochschild cohomology is an isomorphism.

This is an attempt to extend results of [2] where it is shown that the group cohomology $\mathrm{H}^*(G;Z)$ of a torsion-free finitely generated nilpotent group G may be computed using 'numerical cochains'. The extension is two-fold: I consider associative algebras over a commutative noetherian ring k, as a generalization of group algebras over the integers. Secondly, numerical cochains are replaced by the more general cofinite cochains.

The results presented here are preliminary. However, I have tried to write the notes in an elementary and clear fashion so that anyone (including myself) interested in developing the theory further could pick up where I left it.

Contents

1. Preliminaries

Throughout, k will denote a commutative noetherian ring with 1. Unadorned tensor products are over k. The term 'k-algebra' will mean 'associative unitary k-algebra'. If R is a k-algebra then 'R-module' will mean left R-module. We will use without reference standard results of homological algebra, such as those found in [1] or [4]. A k-module M is called *finite* if it is a finitely generated k-module. A submodule M' of a k-module M is called *cofinite in* M (or just *cofinite* when M is clear from the context) if M/M' is finite. In this case we also say that the inclusion is cofinite.

If $\mathcal{A}$ is an abelian category, then $\mathcal{D}^{\geq 0}(\mathcal{A})$ will denote the derived category of non-negative cochain complexes in $\mathcal{A}$. If R is a k-algebra, then $\mathcal{A}^R$ will denote the abelian category of left R-modules and $\mathcal{A}^R_\delta$ will denote the abelian category of δ-modules over R, to be defined below. We denote by $R^e = R \otimes R^{op}$ the enveloping k-algebra of R.

Proposition 1.1. *The intersection of two cofinite submodules is cofinite.*

Proof. Let I, J be cofinite submodules of M. The kernel of the map $M \to M/I \oplus M/J$ sending x to $(x + I, x + J)$ is $I \cap J$, so the map factors through an injection of $M/I \cap J$ into $M/I \oplus M/J$. Since the latter is finite, so is $M/I \cap J$. Here we of course use that k is noetherian. □

Proposition 1.2. *If $M_1 \subseteq M_2 \subseteq M_3$ are inclusions of k-modules then the inclusion $M_1 \subseteq M_3$ is cofinite if and only if both the inclusions $M_1 \subseteq M_2$ and $M_2 \subseteq M_3$ are.*

Proof. This follows from the short exact sequence

$$0 \longrightarrow M_2/M_1 \longrightarrow M_3/M_1 \longrightarrow M_3/M_2 \longrightarrow 0$$

□

Proposition 1.3. *Let R be a k-algebra.*

- *If $M' \subseteq M$ is a cofinite inclusion of R-modules then there is a cofinite two-sided ideal $I \subseteq R$ such that $IM \subseteq M'$.*
- *If M is a finitely generated R-module then the inclusion $IM \subseteq M$ is cofinite for any cofinite ideal $I \subseteq R$.*
- *In particular, any cofinite left or right ideal in R contains a cofinite two-sided ideal.*

Proof. Let M'' be the k-finite R-module M/M'. There is a surjection of k-modules $k^n \to M''$ for some n, so we get an embedding $\mathrm{Hom}_k(M'', M'') \to \mathrm{Hom}_k(k^n, M'') \cong (M'')^n$, showing that $\mathrm{Hom}_k(M'', M'')$ is k-finite. Let $I \subseteq R$ be the kernel of the homomorphism of k-algebras $R \to \mathrm{Hom}_k(M'', M'')$ sending $r \in R$ to the endomorphism $x \mapsto rx$ of M''. Then I is a two-sided ideal and the induced injection of k-modules $R/I \to \mathrm{Hom}_k(M'', M'')$ shows that I is cofinite.

If M is finitely generated there is a surjection of R-modules $R^n \to M$ for some n. If I is a cofinite ideal of R then we get a surjection $(R/I)^n = R/I \otimes_R R^n \to R/I \otimes M \cong M/IM$, which exhibits M/IM as a k-finite module. □

2. δ-maps

Let R be a k-algebra.

Definition 2.1. Let N be a k-module. A k-linear map $f\colon R \to N$ is called a *δ-map* if it vanishes on some two-sided cofinite ideal of R. Equivalently, f is a δ-map if it factors through a homomorphism of k-algebras $\phi\colon R \to S$ where S is k-finite

$$\begin{array}{ccc} R & \xrightarrow{\quad f \quad} & M \\ & {\scriptstyle\phi}\searrow \quad \nearrow & \\ & S & \end{array}$$

The set of δ-maps from R to N is denoted by $\operatorname{Hom}_\delta(R, N)$. Given a k-linear map $f\colon N \to M$, composition with f from the left takes δ-maps to δ-maps, so $\operatorname{Hom}_\delta(R, -)$ is a subfunctor of the functor $\operatorname{Hom}_k(R, -)$ on the category of k-modules.

Maximal cofinite ideals. Let N be a k-module and let $f\colon R \to N$ be a k-linear map. For $r, s \in R$, sfr is the map $(sfr)(x) = f(rxs)$. The k-module

$$J_f = \bigcap_{r,s \in R} \operatorname{Ker}(sfr)$$

is a two-sided ideal contained in $\operatorname{Ker} f$. If f vanishes on a two-sided cofinite ideal I, then certainly $I \subseteq J_f$, so J_f contains all cofinite two-sided ideals contained in $\operatorname{Ker} f$. Therefore we have the following

Proposition 2.2. *$f\colon R \to N$ is a δ-map if and only if J_f is cofinite, and in this case J_f is the unique maximal cofinite two-sided ideal contained in* $\operatorname{Ker} f$.

If R and S are k-algebras, then $R \otimes S$ is a k-algebra by $(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$. In particular we can form the algebra $R^{\otimes n} = R \otimes \ldots \otimes R$ (n factors). Let $\iota_m\colon R \to R^{\otimes n}$ be the natural homomorphism of k-algebras sending r to $1 \otimes \ldots \otimes r \otimes \ldots \otimes 1$ (r at the m^{th} factor). If $\phi\colon R \to R'$ and $\psi\colon S \to S'$ are homomorphisms of k-algebras, then there is an induced homomorphism of k-algebras $\phi \otimes \psi\colon R \otimes S \to R' \otimes S'$, mapping $a \otimes b$ to $\phi(a) \otimes \psi(b)$.

Proposition 2.3. *Let R and S be k-algebras. Any cofinite ideal in $R \otimes S$ contains an ideal of the form $I \otimes S + R \otimes J$, where I and J are cofinite ideals in R and S, respectively.*

Proof. Let $I \subseteq R \otimes S$ be a cofinite ideal. Then $I_R = i_R^{-1}(I)$ and $I_S = i_S^{-1}(I)$, where i_R and i_S are the natural homomorphisms $R, S \to R \otimes S$, are ideals in R and S, respectively. The induced maps

$$R/I_R \longrightarrow R \otimes S/I \longleftarrow S/I_S$$

are injective, which shows that I_R and I_S are cofinite. Clearly, the ideal $I_R \otimes S + R \otimes I_S$ is contained in I. ☐

Proposition 2.4. *A k-linear map $f\colon R^{\otimes n} \to N$ is a δ-map if and only if it factors as*

$$\begin{array}{ccc} R^{\otimes n} & \xrightarrow{\quad f \quad} & N \\ & {\scriptstyle\phi}\searrow \quad \nearrow & \\ & S^{\otimes n} & \end{array}$$

where S is a k-finite algebra and ϕ is induced by a surjective homomorphism of algebras $R \to S$.

Proof. Clearly, the condition is sufficient. To show necessity, factor f as $R^{\otimes n} \to Q \to N$, where Q is a k-finite quotient of $R^{\otimes n}$. Using the previous proposition and induction, the kernel of the surjection $R^{\otimes n} \to Q$ contains an ideal of the form $\sum_{i+j=n-1} R^{\otimes i} \otimes I_i \otimes R^{\otimes j}$ where $I_1, \ldots, I_n \subseteq R$ are cofinite ideals. Then $J = I_1 \cap \ldots \cap I_n$ is a cofinite ideal and $R^{\otimes n} \to Q$ factors as $R^{\otimes n} \to S^{\otimes n} \to Q$, where $S = R/J$. ☐

3. δ-modules

Let R be a k-algebra.

Proposition 3.1. *The following are equivalent for an R-module M:*

- *M is a filtered colimit of k-finite R-modules.*
- *Every cyclic submodule of M is k-finite.*
- *Every R-finite submodule of M is k-finite.*
- *The annihilator of each finite k-submodule of M is a cofinite ideal.*

Proof. The second and third are equivalent since an R-finite submodule is a finite sum of cyclic submodules. If N is a finite k-submodule of M, say generated by $x_1, \ldots, x_n$, then $\operatorname{Ann} N = \operatorname{Ann} x_1 \cap \ldots \cap \operatorname{Ann} x_n$ is cofinite because each $\operatorname{Ann} x_i$ is, as $R/\operatorname{Ann} x_i \cong Rx_i$. The remaining implications are left to the reader. □

Definition 3.2. An R-module M is called a *δ-module over R* if it satisfies the conditions of Proposition 3.1.

If R is a k-algebra then let $\mathcal{A}^R$ denote the abelian category of left R-modules. Let $\mathcal{A}^R_\delta$ denote the full subcategory of $\mathcal{A}^R$ whose objects are the δ-modules.

Proposition 3.3. *$\mathcal{A}^R_\delta$ is a cocomplete abelian category. Furthermore, if R is noetherian then $\mathcal{A}^R_\delta$ is a Serre subcategory of $\mathcal{A}^R$.*

Proof. As $\mathcal{A}^R_\delta$ by definition is a full subcategory of an abelian category, it suffices to check that submodules, quotients and direct sums of δ-modules are δ-modules. Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence of R-modules. Suppose that M is a δ-module. Then for any $x \in M'$, we have that $Rx \cong Rf(x) \subseteq M$ is k-finite, so M' is a δ-module. If $y \in M''$, let $g(x) = y$. Then Rx is k-finite and the surjection $g\colon Rx \to Ry$ shows that Ry is k-finite.

Let $\{M_i\}_{i\in I}$ be a family of δ-modules. If $x \in \oplus_{i\in I} M_i$ then $x = \sum_{i\in J} x_i$, where $x_i \in M_i$, for some finite subset $J \subseteq I$. Therefore, Rx is a submodule of $\oplus_{i\in J} Rx_i$. The latter, being a finite direct sum of k-finite R-modules, is k-finite, so Rx must be k-finite. Hence $\oplus_{i\in I} M_i$ is a δ-module.

Suppose now that R is noetherian. We need to show that if M' and M'' are δ-modules in the short exact sequence above, then so is M. Let $x \in M$. There is a short exact sequence

$$0 \longrightarrow Rx \cap M' \longrightarrow Rx \longrightarrow Rg(x) \longrightarrow 0$$

Here $Rg(x)$ is k-finite as $g(x) \in M''$ and M'' is a δ-module. The R-module $Rx \cap M'$ is a submodule of the finitely generated module Rx and is therefore itself finitely generated, since R is assumed to be noetherian. But it is also a submodule of the δ-module M', so it must then be k-finite. Thus, Rx is an extension of k-finite modules and is therefore k-finite. This proves that M is a δ-module. □

Definition 3.4. Let M be an R-module. Define

$$M_\delta = \{x \in M \mid Rx \text{ is } k\text{-finite}\}.$$

Clearly, M_δ is an R-submodule of M and it is the largest δ-submodule of M. It is the union of all k-finite R-submodules of M. If $f\colon M \to N$ is a map of R-modules, then the induced map $Rx \to Rf(x)$ is surjective, so $f(x) \in N_\delta$ if $x \in M_\delta$. In other words, f restricts to a map of δ-modules $M_\delta \to N_\delta$, so we can regard $(-)_\delta$ as a functor from $\mathcal{A}^R$ to $\mathcal{A}^R_\delta$.

Proposition 3.5. *The functor*

$$(-)_\delta\colon \mathcal{A}^R \to \mathcal{A}^R_\delta$$

is right adjoint to the exact inclusion functor

$$\iota_R\colon \mathcal{A}^R_\delta \to \mathcal{A}^R.$$

Proof. This amounts to the fact that if M is a δ-module and N an R-module, then any map of R-modules $f\colon M \to N$ factors through N_δ. □

Corollary 3.6. $\mathcal{A}^R_\delta$ *has enough injectives.*

Proof. If I is an injective R-module, then I_δ is an injective object of $\mathcal{A}^R_\delta$ because the functor $\operatorname{Hom}_{\mathcal{A}^R_\delta}(-, I_\delta) \cong \operatorname{Hom}_R(\iota_R(-), I)$ is the composite of two exact functors. Thus, if $M \in \mathcal{A}^R_\delta$ then an embedding of M into an injective R-module I gives rise to an embedding of $M = M_\delta$ into the injective object I_δ of $\mathcal{A}^R_\delta$. □

Proposition 3.7. *Suppose that R is noetherian. Then $(M/M_\delta)_\delta = 0$ for any R-module M.*

Proof. In fact, this holds for any right adjoint of an inclusion $\mathcal{A} \to \mathcal{B}$ of a Serre subcategory.

Suppose that N is a δ-submodule of M/M_δ. We need to show that $N = 0$. Let $\pi\colon M \to M/M_\delta$ be the projection and let $L = \pi^{-1}(N)$. Since $\mathcal{A}^R_\delta$ is a Serre subcategory, the exact sequence

$$0 \longrightarrow M_\delta \longrightarrow L \xrightarrow{\ \pi\ } N \longrightarrow 0$$

shows that L is a δ-module. But then applying $(-)_\delta$ to the sequence of inclusions $M_\delta \subseteq L \subseteq M$ yields $L = M_\delta$, which means that $N = 0$. □

One can characterize δ-modules over R as filtered colimits of k-finite R-modules. Moreover, it is obvious that any filtered colimit of δ-modules is a δ-module. In fact, there is yet another way to write δ-modules as filtered colimits. For a two-sided ideal I in R, the forgetful functor

$$\mathcal{A}^{R/I} \to \mathcal{A}^R$$

has a right adjoint,

$$(-)^I\colon \mathcal{A}^R \to \mathcal{A}^{R/I}.$$

If M is an R-module then

$$M^I = \{x \in M \mid Ix = 0\}.$$

This is the largest submodule of M which is a module over R/I. If $I \subseteq J$ then there is an obvious inclusion $M^J \subseteq M^I$.

Proposition 3.8. *Let M be an R-module. Then*

$$M_\delta = \cup_I M^I,$$

where the union is over the set of cofinite two-sided ideals in R. Furthermore,

$$(M_\delta)^I = M^I$$

for any cofinite two-sided ideal $I \subseteq R$.

Proof. If I is a cofinite two-sided ideal in R then $M^I \subseteq M_\delta$, because for any $x \in M^I$ we have $Rx \cong R/\operatorname{Ann}(x)$ which is k-finite as $I \subseteq \operatorname{Ann}(x)$. Conversely, if $x \in M_\delta$ then $\operatorname{Ann}(x)$ is a cofinite left ideal of R. By Proposition 1.3, $\operatorname{Ann}(x)$ contains a cofinite two-sided ideal I, and then $x \in M^I$. □

δ-bimodules. As usual, an R-bimodule is thought of as a module over the k-algebra $R^e = R \otimes R^{op}$. Thus, a δ-bimodule is an object of $\mathcal{A}_\delta^{R^e}$. One can characterize δ-bimodules in terms of their left and right R-module structures.

Proposition 3.9. *The following are equivalent for an R-bimodule M:*

- *M is a δ-bimodule.*
- *M is simultaneously a right and left δ-module over R.*
- $\operatorname{Ann}^b(x) = \{r \in R \mid rx = xr = 0\}$ *is a cofinite k-submodule of R for each $x \in M$.*

Furthermore, the right adjoint $(-)_\delta \colon \mathcal{A}^{R^e} \to \mathcal{A}_\delta^{R^e}$ of the forgetful functor from R-bimodules to δ-bimodules is given by

$$M_\delta = \{x \in M \mid Rx \text{ and } xR \text{ are } k\text{-finite}\}$$

Proof. Any δ-bimodule is a left and right δ-module because Rx and xR are k-submodules of RxR for each $x \in M$, so finiteness of the latter k-module implies finiteness of the former ones as k is assumed noetherian.

Suppose M is a left and right δ-module and let $x \in M$. Then Rx is k-finite, say generated by $x_1, \ldots, x_n \in Rx \subseteq M$. Each x_iR is k-finite, and therefore so is $RxR = x_1R + \ldots + x_nR$. □

For an R-bimodule M and a two-sided ideal I of R, we set

$$M^I = \{x \in M \mid Ix = xI = 0\}.$$

This is a bimodule over R/I and the functor $(-)^I \colon \mathcal{A}^{R^e} \to \mathcal{A}^{(R/I)^e}$ from R-bimodules to R/I-bimodules is right adjoint to the forgetful functor. Furthermore, as in Proposition 3.8 we have

$$M_\delta = \cup_I M^I$$

for bimodules M over R, where the union is over all cofinite two-sided ideals $I \subseteq R$, and we have

$$(M_\delta)^I = M^I$$

for all such I.

4. The bifunctor $\mathrm{Hom}_\delta(M, N)$

The notion of δ-maps may be extended to modules over R.

Definition 4.1. Let M be an R-module and let N be a k-module. A k-linear map $f\colon M \to N$ is called a *δ-map over R*, or simply a *δ-map* if there is no risk of confusion, if f vanishes on some k-cofinite R-submodule of M. The set of δ-maps from M to N will be denoted $\mathrm{Hom}_\delta(M, N)$.

By Proposition 1.3, a k-linear map $R \to N$ vanishes on some left ideal if and only if it does so on some two-sided ideal, so Definition 2.1 is an extension of Definition 4.1.

Note that $\mathrm{Hom}_\delta(M, N)$ is a right R-submodule of $\mathrm{Hom}_k(M, N)$. Indeed, if I is a cofinite submodule of M contained in the kernel of some k-linear map $f\colon M \to N$, then $I \subseteq \mathrm{Ker}\, fr$ for any $r \in R$, and if g is another δ-map that vanishes on a cofinite submodule J , then $\mathrm{Ker}(f + g) \supseteq \mathrm{Ker}\, f \cap \mathrm{Ker} \supseteq I \cap J$, so that $f + g$ is a δ-map.

The next proposition tells us that $\mathrm{Hom}_\delta(-, -)$ may be considered as a bifunctor from $\mathcal{A}^R \times \mathcal{A}^k$ to $\mathcal{A}^{R^{op}}$.

Proposition 4.2. *Suppose given maps $M' \xrightarrow{\phi} M \xrightarrow{f} N \xrightarrow{g} N'$ where ϕ is a map of R-modules, f is a δ-map and g is k-linear. Then gf and $f\phi$ are δ-maps.*

Proof. By assumption, $\mathrm{Ker}\, f$ contains a cofinite submodule I of M. Since $\mathrm{Ker}\, gf \supseteq \mathrm{Ker}\, f \supseteq I$, we see that gf is a δ-map. The R-submodule $\phi^{-1}(I)$ of M' is contained in $\mathrm{Ker}\, f\phi$ and the induced map of R-modules $M'/\phi^{-1}(I) \to M/I$ is injective, which shows that $\phi^{-1}(I)$ is cofinite. □

Proposition 4.3. *For a fixed k-module N, the functor $\mathrm{Hom}_\delta(-, N)\colon \mathcal{A}^{R^{op}} \to \mathcal{A}^R$ is left exact. If N is an injective k-module then $\mathrm{Hom}_\delta(-, N)$ takes cofinite inclusions to surjections.*

Proof. Let $0 \to M' \xrightarrow{\mu} M \xrightarrow{\epsilon} M'' \to 0$ be a short exact sequence of right R-modules. We must show that the sequence

$$0 \longrightarrow \mathrm{Hom}_\delta(M'', N) \xrightarrow{\epsilon^*} \mathrm{Hom}_\delta(M, N) \xrightarrow{\mu^*} \mathrm{Hom}_\delta(M', N)$$

is exact, where $\epsilon^*(f) = f \circ \epsilon$ and $\mu^*(g) = g \circ \mu$. Clearly, $\epsilon^*(f) = 0$ implies $f = 0$ because ϵ is surjective. If g is a δ-map with $\mu^*(g) = 0$, then $g = f \circ \epsilon$ for some $f \in \mathrm{Hom}_k(M'', N)$, since the functor $\mathrm{Hom}_k(-, N)$ is left exact. We must show that f is a δ-map. Let I be a cofinite submodule of M contained in $\mathrm{Ker}\, g$. Then $\epsilon(I)$ is an R-submodule of M'' contained in $\mathrm{Ker}\, f$, and it is cofinite because of the surjection $M/I \to M''/\epsilon(I)$ induced by ϵ.

Next, suppose that N is injective and that the inclusion $M' \to M$ is cofinite, i.e., the quotient M'' is k-finite. Given a δ-map $f\colon M' \to N$ we must produce a δ-map $g\colon M \to N$ that extends f. But N is injective, so we can at least find a k-linear map g extending f. Let $I \subseteq M'$ be a cofinite submodule on which f vanishes. If we assume that $M' \to M$ is cofinite then by transitivity (Proposition 1.2) the composed map $I \to M$ is cofinite. Hence g is a δ-map as it vanishes on the image of I in M. □

Proposition 4.4. *Let M be a finitely presented R-module and let N be a k-module. There is an isomorphism of right R-modules*

$$\mathrm{Hom}_R(M, \mathrm{Hom}_\delta(R, N)) \to \mathrm{Hom}_\delta(M, N),$$

which is natural for maps of finitely presented R-modules.

Proof. The map is defined by sending an R-linear map $f\colon M \to \mathrm{Hom}_\delta(R,N)$ to the map $g\colon M \to N$ given by $g(x) = f(x)(1)$. We need to check that g is indeed a δ-map. Let $x_1,\ldots,x_n$ be R-module generators for M. Each $f(x_i)$ is a δ-map from R to N. Say $f(x_i)$ vanishes on a cofinite ideal I_i. Then $I = I_1 \cap \ldots \cap I_n$ is a cofinite ideal so that IM is a cofinite submodule of M, by Proposition 1.3. Clearly, g vanishes on IM.

The map just defined is clearly natural in M, so we have a natural transformation of contravariant functors from finitely generated R-modules to k-modules

$$\mathrm{Hom}_R(-,\mathrm{Hom}_\delta(R,N)) \to \mathrm{Hom}_\delta(-,N).$$

These functors are both additive and left exact (Proposition 4.3) and they agree on R. Therefore they agree on all finitely presented R-modules. □

Proposition 4.5. *A k-linear map $f\colon R \to N$ is a δ-map if and only if the R-submodule of* $\mathrm{Hom}_k(R,N)$ *generated by f is k-finite. In other words,* $\mathrm{Hom}_\delta(R,N) = \mathrm{Hom}_k(R,N)_\delta$.

Proof. First of all, note that for any left ideal $I \subseteq R$ and any k-linear map $f\colon R \to N$ we have that $I \subseteq \operatorname{Ker} f$ if and only if $I \subseteq \operatorname{Ann} f$. Indeed, $\operatorname{Ann} f \subseteq \operatorname{Ker} f$ is obvious, and for the converse, suppose $I \subseteq \operatorname{Ker} f$ and let $a \in I$. Then for any $x \in R$, $(af)(x) = f(xa) = 0$, since $xa \in I$ as I is a left ideal.

Let $I \subseteq R$ be a cofinite ideal contained in $\operatorname{Ker} f$. Then $I \subseteq \operatorname{Ann} f$, so $\operatorname{Ann} f$ is a cofinite, and therefore $Rf \cong R/\operatorname{Ann} f$ is finite. Conversely, if Rf is finite, then $\operatorname{Ann} f$ is a cofinite left ideal of R contained in $\operatorname{Ker} f$. □

Definition 4.6. A k-algebra R is called *almost finite* if every non-zero ideal in R is cofinite.

If k is a field and R is a Dedekind domain over k, then R is almost finite because the quotient by any non-zero ideal is an artinian k-algebra which is finite dimensional as a k-vector space.

Proposition 4.7. *Suppose that R is an almost finite noetherian k-algebra. Then the functor* $\mathrm{Hom}_\delta(R,-)$ *from $\mathcal{A}^k \to \mathcal{A}^R$ takes injective k-modules to injective R-modules.*

Proof. Let D be an injective k-module. The R-module $E = \mathrm{Hom}_\delta(R,D)$ is injective if and only if $\mathrm{Hom}_R(-,E)$ takes inclusions of left ideal $I \subseteq R$ to surjections. Since R is assumed noetherian all ideals in R are finitely presented, so by Proposition 4.4 we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(R,\mathrm{Hom}_\delta(R,D)) & \xrightarrow{\cong} & \mathrm{Hom}_\delta(R,D) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_R(I,\mathrm{Hom}_\delta(R,D)) & \xrightarrow{\cong} & \mathrm{Hom}_\delta(I,D) \end{array}$$

This shows that $\mathrm{Hom}_\delta(R,D)$ is injective if and only if a surjection $\mathrm{Hom}_\delta(R,D) \to \mathrm{Hom}_\delta(I,D)$ is induced when $I \to R$ is the inclusion of an ideal into R. By 4.3, $\mathrm{Hom}_\delta(-,D)$ takes cofinite inclusions to surjections. We assume that R is almost finite, i.e., that all non-zero ideals in R are cofinite, so it follows that $\mathrm{Hom}_\delta(R,D)$ is an injective R-module. □

Proposition 4.8. *If R is an almost finite noetherian k-algebra then any R-module M may be embedded into an injective R-module E such that E_δ is also injective as an R-module.*

Proof. The functor $\mathrm{Hom}_k(R,-)$ from $\mathcal{A}^k$ to $\mathcal{A}^R$ is right adjoint to the exact forgetful functor $\mathcal{A}^R \to \mathcal{A}^k$, so it preserves injectives. The left R-module structure on $\mathrm{Hom}_k(R,N)$ is given by $rf(s) = f(sr)$. Let $i\colon M \to D$ be an injective k-linear map where D is an injective k-module. Then there is an embedding of R-modules

$$M \xrightarrow{g} \mathrm{Hom}_k(R,M) \xrightarrow{i^*} \mathrm{Hom}_k(R,D)\ ,$$

where for $x \in M$, the k-linear map $g(x)\colon R \to M$ is defined by $g(x)(r) = rx$ for $r \in R$. The R-module $E = \mathrm{Hom}_k(R,D)$ is injective, and by Proposition 4.5, the R-module $E_\delta = \mathrm{Hom}_k(R,D)_\delta$ may be identified with $\mathrm{Hom}_\delta(R,D)$, and this is injective by Proposition 4.7. □

Corollary 4.9. *If R is an almost finite noetherian k-algebra, then the inclusion functor*

$$\iota\colon \mathcal{A}^R_\delta \to \mathcal{A}^R$$

preserves injective objects.

Proof. Let I be an injective object in $\mathcal{A}^R_\delta$. Embed I into an R-module E such that E and E_δ are both injective R-modules. Since I is a δ-module, I lands inside the δ-module E_δ. The monomorphism $I \to E_\delta$ in $\mathcal{A}^R_\delta$ splits as I is injective in this category, so I is a direct summand of E_δ in $\mathcal{A}^R_\delta$. But as $\iota\colon \mathcal{A}^R_\delta \to \mathcal{A}^R$ is fully faithful and exact, I is also a direct summand of E_δ in $\mathcal{A}^R$. Being a direct summand in an injective R-module, the R-module I is itself injective. □

5. Hochschild cohomology and δ-cohomology

Cosimplicial k-modules. If $A = \{A^n\}_{n\geq 0}$ is a cosimplicial k-module, then its associated cochain complex is the graded k-module A with differential $\partial = \sum_i (-1)^i d^i$. The *normalized cochain complex* is the graded k-module $NA = \{NA^n\}$, where

$$NA^n = \bigcap_{i=0}^{n-1} \mathrm{Ker}(s^i) \subseteq A^n.$$

The cosimplicial identities ensure that NA is preserved by ∂ (however, NA is not necessarily preserved by the individual d^i). Obviously, NA is functorial in A. The inclusion $NA \to A$ is a quasi-isomorphism of cochain complexes. Therefore a map $f\colon A \to B$ of cosimplicial k-modules is a weak equivalence if and only if $Nf\colon NA \to NB$ is a quasi-isomorphism.

The Hochschild cosimplicial k-module of a k-algebra. Let R be a k-algebra and let M be an R-bimodule. The *Hochschild cosimplicial k-module* is the graded k-module $C^*(R;M) = \{\mathrm{Hom}_k(R^{\otimes n}, M)\}_{n\geq 0}$ with coface and codegeneracy maps

$$\begin{aligned}
(d^0 f)(r_0,\dots,r_n) &= r_0 f(r_1,\dots,r_n) \\
(d^i f)(r_0,\dots,r_n) &= f(r_0,\dots,r_{i-1}r_i,\dots,r_n) && (0 < i < n+1) \\
(d^{n+1} f)(r_0,\dots,r_n) &= f(r_0,\dots,r_{n-1})r_n \\
(s^i f)(r_1,\dots,r_{n-1}) &= f(r_1,\dots,r_i,1,r_{i+1},\dots,r_{n-1}) && (0 \leq i \leq n-1)
\end{aligned}$$

By definition, the *Hochschild cohomology of R with coefficients in M*, $\mathrm{H}^*(R;M)$, is the cohomology of the corresponding cochain complex. The normalized cochain complex $NC^*(R;M)$ coincides with the classical normalized Hochschild cochain complex.

δ-cochains. Let $C^*_\delta(R,M)$ denote the graded k-submodule of $C^*(R,M)$ with

$$C^n_\delta(R,M) = \mathrm{Hom}_\delta(R^{\otimes n}, M).$$

Proposition 5.1. *If $f \in \mathrm{Hom}_\delta(R^{\otimes n}, M)$, then $d^i f \in \mathrm{Hom}_\delta(R^{\otimes n+1}, M)$ for $i = 1, 2, \ldots, n-1$ and $s^j f \in \mathrm{Hom}_\delta(R^{\otimes n-1}, M)$ for all j.*

Proof. Note that for $0 < i < n$, $d^i f = f \circ d_i$, where $d_i \colon R^{\otimes n+1} \to R^{\otimes n}$ sends $r_0 \otimes \ldots \otimes r_n$ to $r_0 \otimes \ldots \otimes r_{i-1} r_i \otimes \ldots r_n$. Suppose f factors as $R^{\otimes n} \to S^{\otimes n} \to M$, where the first map is induced by a surjective homomorphism $R \to S$ onto a k-finite algebra S, as in Proposition 2.4. The map d_i is natural in k-algebras, so the diagram

$$\begin{array}{ccccc} R^{\otimes n+1} & \xrightarrow{d_i} & R^{\otimes n} & \xrightarrow{f} & M \\ \downarrow & & \downarrow & \nearrow & \\ S^{\otimes n+1} & \xrightarrow{d_i} & S^{\otimes n} & & \end{array}$$

commutes, and yields a factorization of $d^i f = f \circ d_i$ of the required type.

One proceeds similarly for the codegeneracies by noting that $s^j f = f \circ s_j$, where $s_j \colon R^{\otimes n-1} \to R^{\otimes n}$ is the map, natural in R, sending $r_1 \otimes \ldots \otimes r_{n-1}$ to $r_1 \otimes \ldots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \ldots \otimes r_{n-1}$. □

Proposition 5.2. *Suppose that M is a left δ-module over R. Then $d^0 f$ is a δ-map whenever f is one. Similarly, if M is a right δ-module over R, then $d^n f$ is a δ-map if f is one.*

Proof. Let $J \subseteq R^{\otimes n}$ be a cofinite ideal contained in $\mathrm{Ker}\, f$. The kernel of $d^0 f$ contains the ideal $K = \mathrm{Ann}(\mathrm{Im}\, f) \otimes R^{\otimes n} + R \otimes J$ of $R^{\otimes n+1}$, and

$$R^{\otimes n+1}/K \cong \frac{R}{\mathrm{Ann}(\mathrm{Im}\, f)} \otimes \frac{R^{\otimes n}}{J}.$$

But $\mathrm{Im}\, f$ is a finite k-submodule of M, so as M is a δ-module, $\mathrm{Ann}(\mathrm{Im}\, f)$ is a cofinite ideal, by Proposition 3.1. Therefore, both factors above are k-finite, so K is cofinite.

The second part of the proposition is proved in the same way. □

By Proposition 3.9, a δ-bimodule over R, i.e. a δ-module over $R^e = R \otimes R^{op}$, is the same thing as a bimodule over R which is simultaneously a left and right δ-module over R, so we have the following corollary.

Corollary 5.3. *If M is a δ-bimodule over R, then $C^*_\delta(R,M)$ is a cosimplicial submodule of $C^*(R,M)$.*

Definition 5.4. Let M be a δ-bimodule over R. The *δ-cohomology of R with coefficients in M*, $\mathrm{H}^*_\delta(R,M)$, is the cohomology of the cosimplicial k-module $C^*_\delta(R,M)$,

$$\mathrm{H}^n_\delta(R,M) = \mathrm{H}^n(C^*_\delta(R,M)).$$

The inclusion $C^*_\delta(R,M) \subseteq C^*(R,M)$ induces a map of graded k-modules

$$\mathrm{H}^*_\delta(R,M) \to \mathrm{H}^*(R,M).$$

One might ask under what circumstances this map is an isomorphism.

For $m \in M$, $\partial_0(m)$ is the map $R \to M$ given by $\partial_0(m)(r) = rm - mr$. Since M is a δ-bimodule, $\partial_0(m)$ is always a δ-map. Therefore we always have

$$\mathrm{H}^0_\delta(R, M) = \mathrm{H}^0(R, M) = \{m \in M \mid rm = mr \text{ for all } r \in R\}\,.$$

δ-derivations and H^1. The δ-cocycles of degree 1 are precisely the *δ-derivations*, i.e., the δ-maps $d\colon R \to M$ satisfying

$$d(rs) = rd(s) + d(r)s.$$

The 1-coboundaries are the inner derivations $r \mapsto rm - mr$, and since all these are δ-maps, the map $\mathrm{H}^1_\delta(R, M) \to \mathrm{H}^1(R, M)$ is injective, and it is surjective if and only if all derivations $d\colon R \to M$ are δ-derivations.

Lemma 5.5. *A derivation $d\colon R \to M$ is a δ-derivation if and only if the k-module* $\operatorname{Im} d$ *is finitely generated.*

Proof. Let $d\colon R \to M$ be a derivation. Clearly, if d is δ-map, then $\operatorname{Im} d$ is k-finite. Conversely, since $(sdr)(x) = d(rxs) = d(r)xs + rd(x)s + rxd(s)$ for any $r, s, x \in R$, there is an inclusion of k-modules

$$\operatorname{Ann}^b(\operatorname{Im} d) \cap \operatorname{Ker} d \subseteq J_d,$$

with J_d as in Proposition 2.2. If $\operatorname{Im} d$ is k-finite, then $\operatorname{Ker} d$ is cofinite, and so is $\operatorname{Ann}^b(\operatorname{Im} d)$, because M is a δ-module. Hence J_d is also cofinite. ☐

Proposition 5.6. *Let $\phi\colon R \to S$ be a surjective homomorphism of k-algebras. Suppose that the natural map $\mathrm{H}^1_\delta(R; M) \to \mathrm{H}^1(R; M)$ is an isomorphism for all δ-bimodules M over R. Then $\mathrm{H}^1_\delta(S; M) \to \mathrm{H}^1(S; M)$ is an isomorphism for all δ-bimodules M over S.*

Proof. Let $d\colon S \to M$ be a derivation into a δ-bimodule M over S. By pullback along ϕ, M is a δ-modules over R. Hence, by the assumption on R, the derivation $d \circ \phi\colon R \to M$ is a δ-derivation, i.e., $\operatorname{Im} d \circ \phi$ is k-finite. But $\operatorname{Im} d = \operatorname{Im} d \circ \phi$ as ϕ is surjective, so d is a δ-derivation. ☐

Proposition 5.7. *If R is a finitely generated k-algebra then $\mathrm{H}^1_\delta(R, M) \to \mathrm{H}^1(R, M)$ is an isomorphism for any δ-bimodule M.*

Proof. Let $x_1, \ldots, x_n$ be algebra generators for R and let $d\colon R \to M$ be a derivation into a δ-bimodule. We have to show that $\operatorname{Im} d$ is k-finite. The bi-submodule L of M generated by $\operatorname{Im} d$ is finitely generated. Indeed, it is generated by the elements $d(x_1), \ldots, d(x_n)$. Being a sub-bimodule of a δ-bimodule, L is therefore k-finite, which implies that $\operatorname{Im} d \subseteq L$ is k-finite. ☐

Example 5.8. Let $R = k[x_1, x_2, \ldots]$ and let M be the R-module $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the ideal generated by all indeterminates $x_1, x_2, \ldots$. It is easily seen that M is a δ-module. The derivation $d\colon R \to M$ defined by letting $d(x_i)$ be the image of x_i under the projection $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ is not a δ-derivation because its image is not k-finite. This gives an example of a pair (R, M) where $\mathrm{H}^1_\delta(R, M) \to \mathrm{H}^1(R, M)$ is not an isomorphism.

6. Preservation of filtered colimits

Definition 6.1. For any k-algebra R, let Q_R^k denote the opposite category to the category of k-finite algebra quotients of R. The objects are surjective homomorphisms of k-algebras $R \to S$ and a morphism from $R \to S$ to $R \to S'$ is a homomorphism of k-algebras $S' \to S$ such that the diagram below commutes.

$$\begin{array}{ccc} & \nearrow & S' \\ R & & \downarrow \\ & \searrow & S \end{array}$$

By taking kernels, Q_R^k is isomorphic to the set $\mathcal{I}_R^k$ of cofinite two-sided ideals in R partially ordered by reverse inclusion. Sums and intersections of cofinite ideals remain cofinite. In particular the category Q_R^k is both filtered and cofiltered.

Proposition 6.2. *Let N be a k-module and let $n \geq 1$. The natural map*

$$\varinjlim_I \operatorname{Hom}_k((R/I)^{\otimes n}, N) \to \operatorname{Hom}_\delta(R^{\otimes n}, N)$$

is an isomorphism, where the colimit is over the filtered system of cofinite two-sided ideals in R.

Proof. This is merely a reformulation of Proposition 2.4. Namely, a map $f\colon R^{\otimes n} \to N$ is a δ-map if and only if it factors as $R^{\otimes n} \to S^{\otimes n} \to N$, for some k-finite quotient algebra S of R. □

Let M be a δ-bimodule over R. For every inclusion of cofinite two-sided ideals $I \subseteq J$ in R, we have a map $C_\delta^*(R/J, M^J) \to C_\delta^*(R/I, M^I)$ of cosimplicial k-modules obtained as the composite $C_\delta^*(R/J, M^J) \to C_\delta^*(R/I, M^J) \to C_\delta^*(R/I, M^I)$ of the maps induced by the homomorphism $R/I \to R/J$ and the inclusion $M^J \subseteq M^I$ of δ-modules over R/I. This defines a functor $I \mapsto C_\delta^*(R/I, M^I)$ from the filtered system of cofinite two-sided ideals of R to cosimplicial k-modules. By the same token, we have compatible maps $C_\delta^*(R/I, M^I) \to C_\delta^*(R, M)$ and hence an induced map

$$\varinjlim_I C_\delta^*(R/I, M^I) \to C_\delta^*(R, M).$$

Proposition 6.3. *Let M be a δ-bimodule. The canonical map*

$$\varinjlim_I C_\delta^*(R/I, M^I) \to C_\delta^*(R, M)$$

is an isomorphism. The colimit is over the filtered system of cofinite two-sided ideals in R.

Proof. In degree n, the map is the natural one

$$\varinjlim_I \operatorname{Hom}_k((R/I)^{\otimes n}, M^I) \longrightarrow \operatorname{Hom}_\delta(R^{\otimes n}, M)\ .$$

We wish to show that it is an isomorphism.

As in Section 3, the δ-bimodule M is the filtered union $\cup_J M^J$. If $I \subseteq R$ is cofinite, then $(R/I)^{\otimes n}$ is k-finite, so $\operatorname{Hom}_k((R/I)^{\otimes n}, -)$ commutes with filtered colimits. Therefore we have a chain of natural isomorphisms, the first one coming from Proposition 6.2

$$\operatorname{Hom}_\delta(R^{\otimes n}, M) \xleftarrow{\cong} \varinjlim_I \operatorname{Hom}_k((R/I)^{\otimes n}, \cup_J M^J) \xleftarrow{\cong} \varinjlim_I \varinjlim_J \operatorname{Hom}_k((R/I)^{\otimes n}, M^J)$$

The colimits are indexed by the same category. For any category $\mathcal{I}$ the diagonal functor $\mathcal{I} \to \mathcal{I} \times \mathcal{I}$ is cofinal, and thus induces isomorphisms on colimits. Therefore, we can continue our chain of isomorphisms

$$\varinjlim_I \varinjlim_J \mathrm{Hom}_k((R/I)^{\otimes n}, M^J) \xleftarrow{\cong} \varinjlim_I \mathrm{Hom}_k((R/I)^{\otimes n}, M^I)\ .$$

Since all maps in the chain are the natural ones, the composite isomorphism is the natural map

$$\varinjlim_I \mathrm{Hom}_k((R/I)^{\otimes n}, M^I) \to \mathrm{Hom}_\delta(R^{\otimes n}, M).$$

□

Corollary 6.4. *Let M be a δ-bimodule over R. For any $n \geq 0$ the canonical map*

$$\varinjlim_I \mathrm{H}^n(R/I, M^I) \to \mathrm{H}^n_\delta(R, M)$$

is an isomorphism. The colimit is indexed by the filtered system of cofinite two-sided ideals in R.

Proof. We have established an isomorphism of cochain complexes

$$\varinjlim_I C^*(R/I, M^I) \xrightarrow{\cong} C^*_\delta(R, M)\ .$$

The claim follows from the fact that cohomology commutes with filtered colimits.
□

Unlike the ordinary Hom-functor, Hom_δ preserves filtered colimits of k-modules.

Proposition 6.5. *Let $\{N_i\}_{i \in I}$ be a filtered system of k-modules. Then the canonical k-linear map*

$$\varinjlim_i \mathrm{Hom}_\delta(R, N_i) \to \mathrm{Hom}_\delta(R, \varinjlim_i N_i)$$

is an isomorphism.

Proof. Observe that since k is noetherian, any finitely generated k-module S is small in the sense that the canonical map

$$\varinjlim_i \mathrm{Hom}_k(S, N_i) \to \mathrm{Hom}_k(S, \varinjlim_i N_i)$$

is an isomorphism. Then, using Proposition 6.2 one only needs that colimits commute with colimits

$$\begin{aligned}
\varinjlim_i \mathrm{Hom}_\delta(R, N_i) &= \varinjlim_i \varinjlim_S \mathrm{Hom}_k(S, N_i) \\
&= \varinjlim_S \varinjlim_i \mathrm{Hom}_k(S, N_i) \\
&\cong \varinjlim_S \mathrm{Hom}_k(S, \varinjlim_i N_i) \\
&= \mathrm{Hom}_\delta(R, \varinjlim_i N_i)
\end{aligned}$$

□

Corollary 6.6. *Let $\{M_i\}_{i \in I}$ be a filtered system of δ-bimodules over R. Then the canonical map*

$$\varinjlim \mathrm{H}^*_\delta(R, M_i) \to \mathrm{H}^*_\delta(R, \varinjlim M_i)$$

is an isomorphism.

Proof. From the proposition it follows that the canonical map $\varinjlim C^*_\delta(R, M_i) \to C^*_\delta(R, \varinjlim M_i)$ is an isomorphism of cosimplicial k-modules. Since I is filtered, the functor $\varinjlim_i$ is exact, and from this it follows that

$$\varinjlim \mathrm{H}^*_\delta(R, M_i) = \varinjlim \mathrm{H}^*(C^*_\delta(R, M)) \cong \mathrm{H}^*(\varinjlim C^*_\delta(R, M_i)) \cong \mathrm{H}^*_\delta(R, \varinjlim M_i)$$

□

Proposition 6.7. *Let R and S be k-algebras. There is an isomorphism of k-modules*

$$\mathrm{Hom}_\delta(R \otimes S, M) \cong \mathrm{Hom}_\delta(R, \mathrm{Hom}_\delta(S, M))$$

natural in k-modules M.

Proof. It follows from Proposition 2.3 that

$$\mathrm{Hom}_\delta(R \otimes S, M) = \varinjlim_I \varinjlim_J \mathrm{Hom}_k(R/I \otimes S/J, M),$$

where the colimits are over cofinite ideals I and J in R and S respectively. Since R/I is k-finite the functor $\mathrm{Hom}_k(R/I, -)$ commutes with filtered colimits, so we get

$$\begin{aligned} \varinjlim_I \varinjlim_J \mathrm{Hom}_k(R/I \otimes S/J, M) &\cong \varinjlim_I \varinjlim_J \mathrm{Hom}_k(R/I, \mathrm{Hom}_k(S/J, M)) \\ &\cong \varinjlim_I \mathrm{Hom}_k(R/I, \varinjlim_J \mathrm{Hom}_k(S/J, M)) \\ &\cong \mathrm{Hom}_\delta(R, \mathrm{Hom}_\delta(S, M)) \end{aligned}$$

□

Definition 6.8. A k-algebra R is called *nice* if it there is a resolution of R over $R^e = R \otimes R^{\mathrm{op}}$ by finitely presented relatively free R^e-modules.

For example, a noetherian k-algebra is nice.

Proposition 6.9. *Suppose that R is nice. Then for every filtered system $\{M_i\}_{i \in I}$ of R-bimodules the canonical map*

$$\varinjlim \mathrm{H}^*(R, M_i) \to \mathrm{H}^*(R, \varinjlim M_i)$$

is an isomorphism.

Proof. If we compute $\mathrm{H}^*(R, -) = \mathrm{Ext}^*_{R^e/k}(R, -)$ by using a resolution of R by finitely presented relatively free R^e-modules, then the claim follows from the facts that the functor $\mathrm{Hom}_{R^e}(P, -)$ commutes with filtered colimits if P is a finitely presented R^e-module and that homology commutes with filtered colimits. □

7. δ-cohomology as a derived functor

It is useful to know when short exact sequences of coefficient modules give rise to long exact sequences in cohomology. For ordinary Hochschild cohomology, this happens when R is projective as a k-module. The corresponding notion for δ-cohomology is that of a δ-projective algebra.

Definition 7.1. A k-algebra R is called *δ-projective* if for any surjective map of k-modules $f\colon M \to N$, the induced map $f_*\colon \mathrm{Hom}_\delta(R, M) \to \mathrm{Hom}_\delta(R, N)$ is surjective.

Proposition 7.2. *If R and S are δ-projective then so is $R \otimes S$ and R^{op}. In particular, if R is δ-projective, then so is R^e and $R^{\otimes n}$ for all $n \geq 1$.*

Proof. That R is δ-projective means that the functor $\mathrm{Hom}_\delta(R,-)$ is exact. Proposition 6.7 identifies $\mathrm{Hom}_\delta(R\otimes S,-)$ with the composite $\mathrm{Hom}_\delta(R,\mathrm{Hom}_\delta(S,-))$.

There is a natural isomorphism $\mathrm{Hom}_\delta(R^{op},M)\cong\mathrm{Hom}_\delta(R,M)$, because if $I\subseteq R$ is a cofinite two-sided ideal then so is $I^{op}\subseteq R^{op}$ and $R/I\cong R^{op}/I^{op}$ as k-modules. □

Definition 7.3. A k-algebra R is called *strongly δ-projective* if every surjection of k-algebras $R\to S$, where S is k-finite, factors into surjective homomorphisms of k-algebras $R\to Q\to S$ where Q is k-finite and projective as a k-module.

Clearly, strongly δ-projective implies δ-projective. If R is strongly δ-projective, then so is $R^{\otimes n}$. Indeed, any surjection $R^{\otimes n}\to S$ where S is k-finite factors through $Q^{\otimes n}$ for some k-finite projective quotient algebra Q of R, and then $Q^{\otimes n}$ is also k-finite and projective.

Example 7.4. If $p(x)\in k[x]$ is a monic polynomial, then $k[x]/(p(x))$ is a finitely generated free k-module. Also, an ideal $I\subseteq k[x]$ is cofinite if and only if it contains a monic polynomial. Indeed, the sequence of k-submodules $\langle 1\rangle_k\subseteq\langle 1,\alpha\rangle_k\subseteq\langle 1,\alpha,\alpha^2\rangle_k\subseteq\ldots\subseteq k[x]/I$, where $\alpha=x+I$, must stabilize as k is noetherian. Therefore, $\alpha^n=a_{n-1}\alpha^{n-1}+\ldots+a_1\alpha+a_0$ for some $a_i\in k$, so that I contains the polynomial $x^n-a_{n-1}\alpha^{n-1}-\ldots-a_0$.

The polynomial algebra $k[x]$ is strongly δ-projective, because an ideal $I\subseteq k[x]$ is cofinite if and only if it contains a monic polynomial. The quotient of $k[x]$ by such a polynomial is a finitely generated free k-module.

For δ-projective algebras R, we will interpret $\mathrm{H}^n_\delta(R,-)$ as the n^{th} right derived functor $R^n\,\mathrm{H}^0(R,-)$ on the category $\mathcal{A}_\delta^{R^e}$ of δ-bimodules over R.

Proposition 7.5. *Suppose R is δ-projective. Then the functors $\mathrm{H}^n_\delta(R,-)$ form a universal cohomological δ-functor from $\mathcal{A}_\delta^{R^e}$ to $\mathcal{A}^k$.*

Proof. By Proposition 7.2 each $R^{\otimes n}$ is δ-projective. If we have a short exact sequence $0\to M'\to M\to M''\to 0$ of δ-bimodules over R, we therefore get a short exact sequence of cosimplicial k-modules

$$0\to C^*_\delta(R;M')\to C^*_\delta(R;M)\to C^*_\delta(R;M'')\to 0$$

This in turn induces the required long exact sequence in cohomology in the usual way

$$\cdots\to\mathrm{H}^{n-1}_\delta(R;M'')\to\mathrm{H}^n_\delta(R;M')\to\mathrm{H}^n_\delta(R;M)\to\mathrm{H}^n_\delta(R;M'')\to\cdots$$

To prove universality, we show that $\mathrm{H}^n_\delta(R,-)$ is effaceable for every $n\geq 1$. Let M be a δ-bimodule. Embed M in an injective R-bimodule J. Then M is a submodule of J_δ. By the bimodule version of Proposition 3.8 we have $(J_\delta)^I=J^I$, if I is a cofinite two-sided ideal in R. Since $(-)^I\colon\mathcal{A}^{R^e}\to\mathcal{A}^{(R/I)^e}$ is right adjoint to the exact forgetful functor, it takes injectives to injectives. Hence, $(J_\delta)^I=J^I$ is an injective R/I-bimodule for every cofinite two-sided ideal $I\subseteq R$. Therefore, $\mathrm{H}^n(R/I,(J_\delta)^I)=0$ for $n\geq 1$, so by Corollary 6.4

$$\mathrm{H}^n_\delta(R,J_\delta)\cong\varinjlim_I\mathrm{H}^n(R/I,(J_\delta)^I)=0$$

for all $n\geq 1$. □

Remark 7.6. Since $\mathrm{Hom}_\delta(R^{\otimes n}, -)$ is an additive functor it preserves k-split short exact sequences. Therefore, one obtains long exact sequences in δ-cohomology from k-split short exact sequences of δ-bimodules without any assumption on R.

Since $\mathcal{A}_\delta^{R^e}$ has enough injectives by Corollary 3.6, we conclude that $\mathrm{H}_\delta^n(R, -)$ is the n^{th} right derived functor of the restriction of the functor $H^0\colon \mathcal{A}^R \to \mathcal{A}^k$, $M \mapsto \{x \in M \mid rx = xr \text{ for all } r \in R\}$, to the category $\mathcal{A}_\delta^R$. We state this as a proposition.

Proposition 7.7. *Let R be a k-algebra. Consider the follwing diagram of additive functors between abelian categories.*

$$\begin{array}{ccc} \mathcal{A}_\delta^{R^e} & \xrightarrow{\iota_{R^e}} & \mathcal{A}^{R^e} \\ & {\scriptstyle H^0\iota_{R^e}}\searrow \quad \swarrow{\scriptstyle H^0} & \\ & \mathcal{A}^k & \end{array}$$

The right derived functors of H^0 are the Hochschild cohomology functors,

$$R^n(H^0)(M) = \mathrm{H}^n(R; M).$$

If R is δ-projective, then the right derived functors of the restriction $H^0\iota_{R^e}$ of H^0 to $\mathcal{A}_\delta^{R^e}$ are given by

$$R^n(H^0\iota_{R^e})(M) = \mathrm{H}_\delta^n(R; M).$$

In other words, $R^*(H^0\iota_R)$ may be computed as the cohomology of the cochain complex $C_\delta^*(R; M)$ of δ-cochains. Furthermore, the natural transformation $R(H^0\iota_{R^e}) \to R(H^0)R(\iota_{R^e})$ of triangulated functors $\mathcal{D}^{\geq 0}(\mathcal{A}_\delta^{R^e}) \to \mathcal{D}^{\geq 0}(\mathcal{A}^k)$ induces for each δ-bimodule M a map in cohomology

$$\mathrm{H}_\delta^*(R; M) \to \mathrm{H}^*(R; M),$$

which may be identified with the map induced in cohomology by the inclusion of cochain complexes $C_\delta^*(R; M) \to C^*(R; M)$.

Definition 7.8. A k-algebra R is called *stable* if the inclusion functor $\iota_R\colon \mathcal{A}_\delta^R \to \mathcal{A}^R$ preserves injective objects.

Corollary 4.9 says that almost finite noetherian k-algebras are stable. We will see later that any commutative noetherian k-algebra is stable.

Corollary 7.9. *Suppose that R is a δ-projective k-algebra whose enveloping algebra R^e is stable. Then the natural map $\mathrm{H}_\delta^n(R; M) \to \mathrm{H}^n(R; M)$ is an isomorphism for all δ-bimodules M over R.*

8. Change of ground ring

Let $\phi\colon k \to l$ be a homomorphism of commutative rings. Any l-module is a k-module by pullback along ϕ. In particular l is a k-module. If R is a k-algebra, then R_l denotes the l-algebra $l \otimes_k R$. There is a natural homomorphism of k-algebras $j = \phi \otimes 1\colon R = k \otimes_k R \to R_l$. If M is an R-bimodule then $M_l = l \otimes_k M$ is an R_l-bimodule. There is a functor $Q_R^k \to Q_{R_l}^l$ acting on objects in the obvious way: a surjection $R \to S$ is sent to the surjection $R_l \to S_l$. It is a classical result that

$$\mathrm{H}_k^*(R, M) \cong \mathrm{H}_l^*(R_l, M)$$

for any R_l-bimodule M. A natural question is what happens for δ-cohomology.

Proposition 8.1. *Let $k \to l$ be a homomorphism of commutative rings. There is an inclusion of l-modules*

$$\mathrm{Hom}_{\delta,k}(R,N) \subseteq \mathrm{Hom}_{\delta,l}(R_l,N)$$

natural in l-modules N. The following are equivalent:

- $\mathrm{Hom}_{\delta,k}(R,N) = \mathrm{Hom}_{\delta,l}(R_l,N)$ *for all l-modules N.*
- *Every l-cofinite ideal of R_l pulls back to a k-cofinite ideal of R along the natural map $R \to R_l$.*
- *The functor $Q^k_R \to Q^l_{R_l}$ is cofinal.*

Proof. The inclusion is defined by sending a δ_k-map $f\colon R \to N$ to the l-linear map $f_l\colon R_l \to N$ given by $f_l(\lambda \otimes r) = \lambda f(r)$. It is clear that f_l is a δ_l-map because a factorization $R \to S \to N$ of f yields a factorization $R_l \to S_l \to N$ of f_l, and S k-finite implies S_l l-finite. We always have that $f_l \circ j = f$, so it is clear $f \mapsto f_l$ is injective. We have equality if and only if $g \circ j$ is a δ_k-map whenever $g\colon R_l \to N$ is a δ_l-map. Now, in the case of equality, let $I \subseteq R_l$ be an l-cofinite ideal. Then the projection $f\colon R_l \to R_l/I$ is a δ_l-map, so by assumption $g = f \circ j$ is a δ_k-map. Hence $j^{-1}(I) = \mathrm{Ker}\, g$ is k-cofinite in R. Conversely, assume that l-cofinite ideals of R_l pull back to k-cofinite ideals of R. Let $f\colon R_l \to N$ be a δ_l-map. We must show that $f \circ j$ is a δ_k-map, i.e., we need to find a k-cofinite ideal of R on which $f \circ j$ vanishes. But the ideal $j^{-1}(I)$, which is cofinite by assumption, will do.

For the equivalence of the second and third statements, if we interpret Q^k_R as the set of k-cofinite ideals of R partially ordered by reverse inclusion, and similarly for $Q^l_{R_l}$, then the functor $Q^k_R \to Q^l_{R_l}$ is given by mapping a k-cofinite ideal $I \subseteq R$ to the *extension* of I, i.e., the ideal I_l generated by $j(I)$. In this setup, cofinality is equivalent to the statement that every l-cofinite ideal J in R_l contains the extension of some k-cofinite ideal $I \subseteq R$. But if $j^{-1}(J)$ is k-cofinite for every such J, then J contains the extended ideal $(j^{-1}(J))_l$. Conversely, if every l-cofinite J contains the extension of a k-cofinite I, then $j^{-1}(J) \supseteq j^{-1}(I_l) \supseteq I$, implying that $j^{-1}(J)$ is k-cofinite. □

Proposition 8.2. *Suppose that the equivalent conditions of Proposition 8.1 are satisfied and that in addition l is flat as a k-module. Then for any k-module N, there is an isomorphism of l-modules*

$$l \otimes_k \mathrm{Hom}_{\delta,k}(R,N) \to \mathrm{Hom}_{\delta,l}(R_l,N_l).$$

Proof. If S is k-finite, then it is finitely presented as a k-module, and hence the map $l \otimes_k \mathrm{Hom}_k(S,N) \to \mathrm{Hom}_l(S_l,N_l)$ is an isomorphism as l is flat as a k-module. Since tensor products commute with filtered colimits, we have a sequence of isomorphisms

$$\begin{aligned} l \otimes_k \mathrm{Hom}_{\delta,k}(R,N) &\cong l \otimes_k \varinjlim_S \mathrm{Hom}_k(S,N) \cong \varinjlim_S l \otimes_k \mathrm{Hom}_k(S,N) \\ &\cong \varinjlim_S \mathrm{Hom}_l(S_l,N_l) \cong \varinjlim_{S'} \mathrm{Hom}_l(S',N_l) = \mathrm{Hom}_{\delta,l}(R_l,N_l). \end{aligned}$$

Here the colimits are over $S \in Q^k_R$ and $S' \in Q^l_{R_l}$. The second to last isomorphism changing the index category comes from the fact that the functor $Q^k_R \to Q^l_{R_l}$ is cofinal, by Proposition 8.1, and hence induces an isomorphism on colimits. □

Example 8.3.

- The conditions in Proposition 8.1 are fulfilled when l is k-finite.

- The conditions are not fulfilled for $R = \mathbb{Z}[x]$ and $k \to l$ being the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$. For instance, the $\mathbb{Q}$-cofinite ideal of $\mathbb{Q}[x]$ generated by $x - 1/2$ pulls back to the ideal in $\mathbb{Z}[x]$ generated by $2x - 1$, but this ideal is not $\mathbb{Z}$-cofinite.

Proposition 8.4. *Let $k \to l$ be a homomorphism of commutative rings satisfying the conditions of Proposition 8.1. Then for any $n \geq 0$ there is an isomorphism of l-modules $\mathrm{H}^n_{\delta,k}(R, M) \to \mathrm{H}^n_{\delta,l}(R_l, M)$ natural in δ-bimodules M over R_l that fit in a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^n_{\delta,k}(R,M) & \xrightarrow{\cong} & \mathrm{H}^n_{\delta,l}(R_l,M) \\ \downarrow & & \downarrow \\ \mathrm{H}^n_k(R,M) & \xrightarrow{\cong} & \mathrm{H}^n_l(R_l,M) \end{array}$$

Proof. Note that $(R_l)^{\otimes_l n} \cong (R^{\otimes n})_l$. Therefore, we get isomorphisms

$$\mathrm{Hom}_{\delta,k}(R^{\otimes n}, M) \to \mathrm{Hom}_{\delta,l}((R_l)^{\otimes_l n}, M)$$

for all n by Proposition 8.1, and it is clear that these isomorphisms are compatible with the coface and codegeneracy maps, which means that we have an isomorphism of cosimplicial l-modules $C^*_{\delta,k}(R, M) \to C^*_{\delta,l}(R_l, M)$. This isomorphism sits inside a commutative diagram of cosimplicial l-modules

$$\begin{array}{ccc} C^*_{\delta,k}(R,M) & \xrightarrow{\cong} & C^*_{\delta,l}(R_l,M) \\ \downarrow & & \downarrow \\ C^*_k(R,M) & \xrightarrow{\cong} & C_l(R_l,M) \end{array}$$

Now apply cohomology. □

9. δ-cohomology of polynomial algebras

We will show that the Hochschild cohomology $\mathrm{H}^*(k[x], M)$ of the polynomial algebra $k[x]$, with coefficients in any δ-bimodule M, may be computed using δ-cochains, i.e., we will show that the map $\mathrm{H}^*_\delta(k[x], M) \to \mathrm{H}^*(k[x], M)$ is an isomorphism. This will be done by reduction to the case when $M = k$, and in this case by an explicit calculation.

The next proposition follows immediately from Proposition 2.4 and the description of cofinite ideals in $k[x]$.

Proposition 9.1. *Let N be any k-module. There is an isomorphism of k-modules*

$$\mathrm{Hom}_k(k[x_1, \ldots, x_n], N) \cong N[\![z_1, \ldots, z_n]\!]$$

given by mapping a k-linear map $f\colon k[x_1, \ldots, x_n] \to N$ to the series

$$S_f = \sum_{\alpha \in \mathbb{N}^n} f(x^\alpha) z^\alpha.$$

Here $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. Furthermore, f is a δ-map if and only if there is a polynomial $q(z) \in k[z]$ with $q(0) = 1$ such that $q(z_1) \ldots q(z_n) S_f \in N[z_1, \ldots, z_n]$.

A series of the form S_f for some δ-map f will be called a *δ-series*.

We will now study the cosimplicial k-module $C^*_\delta(k[x], k)$, where k is considered a $k[x]$-bimodule via $x \cdot 1 = c_l$ and $1 \cdot x = c_r$ for some $c_l, c_r \in k$. If one traces the coface and codegeneracy maps through the isomorphism of Proposition 9.1 then one gets the following description of $C^*(k[x], k)$:

For compactness of notation, write $P_{i_1 \ldots i_n} = P(z_{i_1}, \ldots, z_{i_n})$ if P is a formal power series in n indeterminates. The component in degree n is the k-module $C^n(k[x], k) = k[\![z_1, \ldots, z_n]\!]$, and the coface and codegeneracy maps

$$d^i \colon C^{n-1}(k[x], k) \to C^n(k[x], k), \quad s^i \colon C^{n+1}(k[x], k) \to C^n(k[x], k)$$

for $0 \leq i \leq n$ are given by

$$\begin{aligned} d^0(S)_{12\ldots n} &= \frac{S_{23\ldots n}}{1 - c_l z_1} \\ d^i(S)_{12\ldots n} &= \frac{z_i S_{12\ldots\widehat{i+1}\ldots n} - z_{i+1} S_{12\ldots\hat{i}\ldots n}}{z_i - z_{i+1}} \\ d^n(S)_{12\ldots n} &= \frac{S_{12\ldots(n-1)}}{1 - c_r z_n} \\ s^i(T)_{12\ldots n} &= T(z_1, \ldots, z_i, 0, z_{i+1}, \ldots, z_n) \end{aligned}$$

Let $\Pi(z_1, \ldots, z_n)$ be the polynomial

$$(1 - c_l z_1)(z_1 - z_2)(z_2 - z_3) \ldots (z_{n-1} - z_n)(1 - c_r z_n).$$

Proposition 9.2. *Let $S \in k[\![z_1, \ldots, z_{n-1}]\!]$. Then*

$$\partial(S)_{12\ldots n} = \frac{\sum_{i=1}^n (-1)^{i-1} z_i (\Pi S)_{12\ldots\hat{i}\ldots n}}{\Pi_{12\ldots n}} \tag{1}$$

Proof. Elementary calculation. □

Proposition 9.3. *The natural map $\mathrm{H}^n_\delta(k[x], k) \to \mathrm{H}^n(k[x], k)$ is an isomorphism for all $n \geq 0$.*

Proof. By Proposition 5.7 the map $\mathrm{H}^i_\delta(k[x], k) \to \mathrm{H}^i(k[x], k)$ is an isomorphism for $i = 0, 1$. For notational convenience, write $A = C^*(k[x], k)$ and $B = C^*_\delta(k[x], k)$.

Clearly, $\mathrm{Ker}(s^i \colon A^n \to A^{n-1})$ is the ideal generated by z_{i+1}, for $i = 0, 1, \ldots, n-1$. Therefore the normalized cochain complex NA of A is in degree n the submodule of series S of the form $S = z_1 \ldots z_n P$ for some series P. The n-cochains S of NB have the same description but with P a δ-series.

Let S be an $(n-1)$-cocycle of NA, where $n \geq 3$, say $S = z_1 \ldots z_{n-1} P$. Then from (1) we see that

$$\begin{aligned} 0 = (\Pi \partial S)_{12\ldots n} &= \sum_{i=1}^n (-1)^{i-1} z_i (\Pi S)_{12\ldots\hat{i}\ldots n} \\ &= \sum_{i=1}^n (-1)^{i-1} z_1 \ldots z_n (\Pi P)_{12\ldots\hat{i}\ldots n}, \end{aligned}$$

which is equivalent to

$$\sum_{i=1}^n (-1)^{i-1} (\Pi P)_{12\ldots\hat{i}\ldots n} = 0.$$

Setting $z_n = 0$ in this equality of formal power series, we obtain

$$\sum_{i=1}^{n-1}(-1)^{i-1}(\Pi P)(z_1,\ldots,\hat{z_i},\ldots,z_{n-1},0)+(-1)^{n-1}(\Pi P)(z_1,\ldots,z_{n-1})=0,$$

and multiplying this with $z_1 \ldots z_{n-1}$ we get

$$\begin{aligned}(-1)^n(\Pi S)_{12\ldots(n-1)} &= \sum_{i=1}^{n-1}(-1)^{i-1}z_1\ldots z_{n-1}(\Pi P)(z_1,\ldots,\hat{z_i},\ldots,z_{n-1},0)\\ &= \sum_{i=1}^{i-1}(-1)^{i-1}z_iQ_{12\ldots\hat{i}\ldots(n-1)}\\ &= (\Pi\partial Q)_{12\ldots(n-1)},\end{aligned}$$

where $Q(z_1,\ldots,z_{n-2}) = z_1\ldots z_{n-2}(\Pi P)(z_1,\ldots,z_{n-2},0)$. Hence $S = \partial((-1)^nQ)$ is a coboundary. We have now shown by hand that $\mathrm{H}^n(NA) = 0$ for $n \geq 2$. This is of course no surprise and it could be shown in a few lines. The point however is the explicit description of the cochain $(-1)^nQ$ whose coboundary is the given cocycle S. The apparent but crucial observation is the following: If $S \in NB^{n-1}$, then

$$P(z_1,\ldots,z_{n-1}) = \frac{p(z_1,\ldots,z_{n-1})}{q(z_1)\ldots q(z_{n-1})},$$

for polynomials p, q with coefficients in k and $q(0) = 1$, and it follows that

$$Q(z_1,\ldots,z_{n-2}) = z_1\ldots z_{n-2}\frac{(\Pi p)(z_1,\ldots,z_{n-2},0)}{q(z_1)\ldots q(z_{n-2})},$$

so that $Q \in NB^{n-1}$. Therefore, we see that $\mathrm{H}^n(NB) = 0$ for $n \geq 2$. We conclude that the inclusion $B \to A$ induces an isomorphism in cohomology. □

Remark 9.4. Actually, the graded subspace $C = \{k[z_1,\ldots,z_n]\}_{n\geq 0}$ of A is preserved by the coface and codegeneracy maps, and one sees that if S is a polynomial $(n-1)$-cocycle then the cochain $(-1)^nQ$ whose coboundary is S is also a polynomial. So the inclusion $C \to A$ is a weak equivalence by the same argument.

Proposition 9.5. *Let M be any δ-bimodule over $k[x]$. The natural map*

$$\mathrm{H}^n_\delta(k[x],M) \to \mathrm{H}^n(k[x],M)$$

is an isomorphism for all n.

Proof. We will use the machinery developed so far to reduce to the case when $M = k$.

The δ-module M is a filtered colimit, $\varinjlim_i M_i$, of k-finite $k[x]$-bimodules M_i. For each $n \geq 0$, we have a commutative diagram

$$\begin{array}{ccc}\varinjlim_i \mathrm{H}^n_\delta(k[x],M_i) & \longrightarrow & \mathrm{H}^n_\delta(k[x],M)\\ \downarrow & & \downarrow\\ \varinjlim_i \mathrm{H}^n(k[x],M_i) & \longrightarrow & \mathrm{H}^n(k[x],M)\end{array}$$

The top horizontal map is an isomorphism by Corollary 6.6 and since $k[x]$ is nice, the bottom map is also an isomorphism. Therefore the right map is an isomorphism if and only if the left one is. But this is induced by the natural maps $\mathrm{H}^n_\delta(k[x],M_i) \to \mathrm{H}^n(k[x],M_i)$. Thus, we have reduced to the case when M is a k-finite R-bimodule.

If M is k-finite, then M is certainly finitely generated as an $k[x]$-bimodule. Let $m(M)$ denote the minimal number of bimodule generators for M. Suppose $m(M) = r$ and let $x_1, \dots, x_r$ be bimodule generators for M. Let N be the bisubmodule of M generated by x_r. Then we have a short exact sequence of $k[x]$-bimodules

$$0 \to N \to M \to M/N \to 0$$

where N and M/N are k-finite, $m(N) = 1$, and $m(M/N) \leq r - 1$, since M/N can be generated by the images of $x_1, \dots, x_{r-1}$ in M/N.

Since $k[x]$ is both projective and strongly δ-projective, any short exact seqences of $k[x]$-bimodules $0 \to M' \to M \to M'' \to 0$ gives rise to a ladder with exact rows

$$\begin{array}{ccccccccc} \cdots \to & \mathrm{H}_\delta^{n-1}(k[x], M'') & \to & \mathrm{H}_\delta^n(k[x], M') & \to & \mathrm{H}_\delta^n(k[x], M) & \to & \mathrm{H}_\delta^n(k[x], M'') & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & \mathrm{H}^{n-1}(k[x], M'') & \to & \mathrm{H}^n(k[x], M') & \to & \mathrm{H}^n(k[x], M) & \to & \mathrm{H}^n(k[x], M'') & \to \cdots \end{array}$$

It follows from the 5-lemma that if the maps from δ-cohomology to cohomology with coefficients in M' and M'' are isomorphisms, then so are the maps $\mathrm{H}_\delta^n(k[x], M) \to \mathrm{H}^n(k[x], M)$.

Therefore, by induction on $m(M)$, we may reduce to the case $m(M) = 1$, i.e., to the case of k-finite cyclic $k[x]$-bimodules. A bimodule over $k[x]$ is the same thing as a left module over $k[x] \otimes k[x]^{\mathrm{op}} \cong k[x, y]$, so a k-finite cyclic $k[x]$-bimodule is of the form $M = k[x, y]/I$ for some cofinite ideal $I \subseteq k[x, y]$, where x acts as multiplication by x from the left and multiplication by y from the right. Now for the twist. Not only is $k[x, y]/I$ a $k[x]$-bimodule, but it is also a commuative noetherian k-algebra, which we may denote by l. Now, l is an $l[x]$-bimodule by letting x act by multiplication by α to the left and by multiplication by β to the right, where $\alpha = x + I \in l$ and $\beta = y + I \in l$. Moreover, the $l[x]$-bimodule l is pulled back to the $k[x]$-bimodule $k[x, y]/I$ along the homomorphism $k[x] \to l[x]$. Since l is k-finite, we have by Proposition 8.1 that the ring extension $k \to l$ induces an isomorphism

$$\mathrm{H}_{\delta,k}^n(k[x], M) \cong \mathrm{H}_{\delta,l}^n(l[x], l)$$

for all $n \geq 0$. Also, the base change $k \to l$ induces an isomorphism in ordinary cohomology $\mathrm{H}_k^n(k[x], M) \to \mathrm{H}_l^n(l[x], l)$ and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_{\delta,k}^n(k[x], M) & \xrightarrow{\cong} & \mathrm{H}_{\delta,l}^n(l[x], l) \\ \downarrow & & \downarrow \\ \mathrm{H}_k^n(k[x], M) & \xrightarrow{\cong} & \mathrm{H}_l^n(l[x], l) \end{array}$$

The right vertical map is an isomorphism by Proposition 9.3, so it follows that the left map is an isomorphism too. □

We will now prove a similar result for $k[x, x^{-1}]$. Since $k[x, x^{-1}]$ is the group algebra of $\mathbb{Z}$, this can be interpreted as saying that the cohomology of the additive group $\mathbb{Z}$ may be computed using δ-cochains.

Proposition 9.6. *Let M be any δ-bimodule over $k[x, x^{-1}]$. The natural map $\mathrm{H}_\delta^n(k[x, x^{-1}], M) \to \mathrm{H}^n(k[x, x^{-1}], M)$ is an isomorphism for all n.*

Proof. The δ-bimodule M pulls back to a δ-bimodule over $k[x]$ via the canonical homomorphism $k[x] \to k[x, x^{-1}]$. It is classical, or in any case not hard to show, that this homomorphism induces an isomorphism $\mathrm{H}^n(k[x, x^{-1}], M) \to \mathrm{H}^n(k[x], M)$ for all n. According to Proposition 9.5 the natural map $\mathrm{H}^n_\delta(k[x], M) \to \mathrm{H}^n(k[x], M)$ is an isomorphism.

The map $C^*_\delta(k[x, x^{-1}], M) \to C^*_\delta(k[x], M)$ is an isomorphism. Indeed, an ideal $I \subseteq k[x, x^{-1}]$ is cofinite if and only if it contains a 'bimonic' polynomial, that is, a polynomial of the form $x^{r+1} + c_r x^r + \ldots + c_1 x + 1$. Therefore, $f\colon k[x, x^{-1}]^{\otimes n} \to M$ is a δ-map if and only if one can find a bimonic polynomial $p(x)$ such that $f(q_1(x), \ldots, q_n(x)) = 0$ whenever some $q_i(x)$ can be written as $q_i(x) = p(x)s(x)$ for some $s(x) \in k[x, x^{-1}]$. From this it follows that f is determined by its values on $x^{a_1} \otimes \ldots \otimes x^{a_n}$ for $a_i \in \{0, 1, \ldots, r\}$. In particular, f is determined by its restriction to $k[x]^{\otimes n}$, and for similar reasons it is clear that any δ-map $f\colon k[x]^{\otimes n} \to M$ extends to $k[x, x^{-1}]^{\otimes n}$. This means that the map $C^*_\delta(k[x, x^{-1}], M) \to C^*_\delta(k[x], M)$ is bijective.

The claim now follows by passing to cohomology in the commutative diagram

$$\begin{array}{ccc} C^*_\delta(k[x, x^{-1}], M) & \longrightarrow & C^*_\delta(k[x], M) \\ \downarrow & & \downarrow \\ C^*(k[x, x^{-1}], M) & \longrightarrow & C^*(k[x], M) \end{array}$$

□

10. The cofinite topology

In this section we will rely on results proved in [3]. See also [5].

Let R be a k-algebra. The set $\mathcal{I}^k_R$ of cofinite ideals in R forms a fundamental system of neighborhoods of 0 for a linear topology on R, which we will call the *cofinite topology*. An R-module M is topologized by letting the open neighborhoods of 0 be the submodules $L \subseteq M$ such that M/L is a δ-module. A module is discrete in this topology if and only if it is a δ-module. Proposition 3.3 implies that the cofinite topology is a Gabriel topology (cf. [5]) provided R is noetherian. Proposition 1.3 implies that the cofinite topology is *bounded*, i.e., that it has a basis consisting of two-sided ideals.

Recall that a k-algebra R is stable if the inclusion functor $\iota_R\colon \mathcal{A}^R_\delta \to \mathcal{A}^R$ preserves injective objects. As a consequence of the identification of δ-modules as the discrete modules for a topology on R, we get a characterization of stable k-algebras as follows, cf. [3] Proposition V.9.

Proposition 10.1. *A k-algebra R is stable if and only if for every R-module M, the subspace topology on every submodule $M' \subseteq M$ coincides with the cofinite topology on M'.*

Concretely, the last condition means that whenever we have inclusions of R-modules $L' \subseteq M' \subseteq M$ such that M'/L' is a δ-module there is a submodule $L \subseteq M$ such that M/L is a δ-module and $L \cap M' = L'$.

Proposition 10.2. *Commutative noetherian k-algebras are stable.*

Proof. If R is noetherian, then the cofinite topology is a bounded Gabriel topology. According to [3] Proposition V.10, any bounded Gabriel topology on a commutative noetherian ring is stable, so in particular R is stable for the cofinite topology. □

Since polynomial algebras are δ-projective, the next corollary subsumes the results of the previous section.

Corollary 10.3. *If R is a δ-projective commutative noetherian k-algebra, then the natural map $\mathrm{H}^n_\delta(R;M) \to \mathrm{H}^n(R;M)$ is an isomorphism for all $n \geq 0$ and all δ-bimodules M over R.*

Proof. The enveloping algebra R^e of a commutative noetherian algebra R is still commutative and noetherian and hence stable. The claim now follows from Corollary 7.9. □

References

[1] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press 1956

[2] T. Ekedahl, *On minimal models in integral homotopy theory*, The Roos Festschrift, Homology Homotopy Appl. **4** (2002), 191–218

[3] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448

[4] S. Mac Lane *Homology*, Springer-Verlag 1975

[5] B. Stenström, *Rings of quotients*, Die Grundlehren der Mathematischen Wissenschaften, Band 217. Springer-Verlag 1975