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Veronica Crispin Quiñonez

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Department of Mathematics
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Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

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Veronica Crispin Quiñonez
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden
e-mail: veronica@math.su.se

1 Introduction

For an ideal I in a ring R a reduction is defined as an ideal $J \subseteq I$ such that $JI^l = I^{l+1}$ for some integer l. An ideal which is minimal with respect to this property is called *minimal reduction*, and the least integer l for these ideals is called the *reduction number* of I. Further, J is a reduction of I if and only if I is integral over J.

To a monomial ideal I in $k[x_1, \dots, x_n]$ or $k[[x_1, \dots x_n]]$ we associate the set $\log I$ which consists of the exponents of all monomials belonging to I, that is, $\log I = \{(a_1, \dots a_n) \mid x_1^{a_1} \cdots x_n^{a_n} \in I\}$. It is proved ([2], [3], [6]) that the integral closure of I is the integrally closed monomial ideal, $\bar{I}$, generated by all the monomials with exponents in the convex hull of $\log I$ in $\mathbb{N}^n$. In this paper we determine minimal reductions of any monomial ideal in the ring $k[[x, y]]$ using the relation between reduction and integral closure.

2 Monomial ideals

Consider the two-dimensional local ring $R = k[[x, y]]$ and an ideal of dimension two in it. According to the general theory for minimal reductions, a minimal reduction of the ideal will be generated by two elements in the ring, if the residue field is infinite. However, the results that will follow are valid for any field k in $k[[x, y]]$. Any monomial ideal I in R is of the form $I = mI'$, where m is a monomial and I' an $\mathfrak{m}$-primary monomial ideal. An ideal mJ is a reduction of mI' if and only if J is a reduction of I'. Thus, we may assume that I is $\mathfrak{m}$-primary, that is, that x^a and y^b belong to I for some a and b.

For a moment we may let the ring R be the polynomial ring also. Let $I = \langle x^{A_i} y^{B_i} \rangle_{0 \le i \le s} \subset k[x, y]$ or $k[[x, y]]$ where $A_i < A_{i+1}$, $B_i > B_{i+1}$ and $A_0 = B_s = 0$. We will construct a monomial ideal, which we call I_{lmr}, having the same integral closure as I, and show that it is the unique smallest one. In other words, $\text{conv}(\log I_{lmr}) = \log \bar{I}$.

Definition 2.1. Let $I = \langle x^{A_i} y^{B_i} \rangle$. Define $I_{lmr} = \langle x^{A_{i_j}} y^{B_{i_j}} \rangle$ as follows:

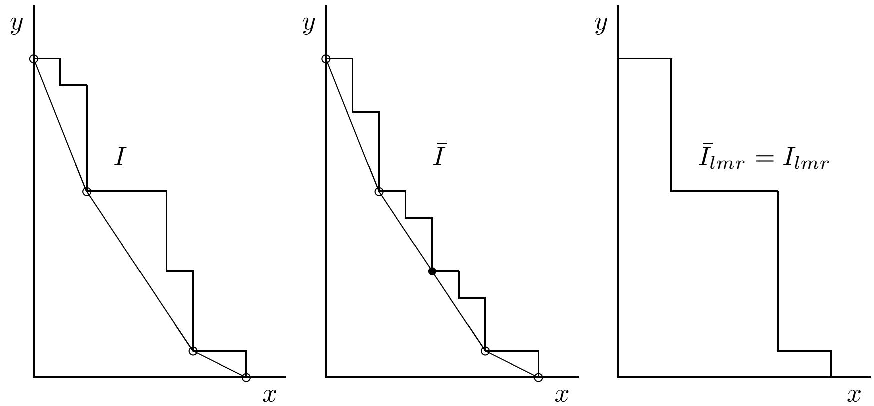
$i_0 = 0$,

i_1 be the greatest i such that the minimal value of the expression $\frac{A_i}{B_0 - B_i}$ is obtained,

for $j \geq 2$ let $i_j = \max \{ i > i_{j-1};\ \frac{A_i - A_{i_{j-1}}}{B_{i_{j-1}} - B_i}$ is minimal $\}$.

Graphically we define the generators of I_{lmr} in $\mathbb{N}^2$ recursively by starting with $(0, B_0)$ and choosing the greatest index i such that (A_i, B_i) gives the steepest slope between the two points. Taking this exponent as our new starting point we repeat the procedure.

Example 2.2. Let $I = \langle y^{12}, xy^{11}, x^2y^7, x^5y^4, x^8 \rangle$. Then the integral closure is $\bar{I} = \langle y^{12}, xy^{10}, x^2y^7, x^3y^6, x^4y^4, x^5y^3, x^6y, x^8 \rangle$ and the least monomial reduction of both ideals is $I_{lmr} = \langle y^{12}, x^2y^7, x^6y, x^8 \rangle$, the generators of which are marked by empty circles.



It is clear that among all monomial ideals with integral closure $\bar{I}$ the ideal I_{lmr} is the least one. Equivalently, among all monomial ideals which are reductions of $\bar{I}$, the ideal I_{lmr} is least. We call it *least monomial reduction*. Moreover, I_{lmr} is the least monomial reduction of any ideal lying between itself and $\bar{I}$.

Remark 2.3. Any integrally closed monomial ideal is a product of *blocks* (Theorem 3.8 in [2]), where we by an (a, b)-block mean the unique simple integrally closed monomial ideal containing the elements x^a and y^b in its minimal generating set, that is, $\overline{\langle x^a, y^b \rangle}$ where a and b are relatively prime. (We recall that an ideal is called simple if it cannot be written as a product of two proper ideals.) The product of s number of (a_i, b_i)-blocks satisfying the condition $\frac{a_i}{b_i} \leq \frac{a_{i+1}}{b_{i+1}}$ is the integrally closed ideal

$$\prod_{i=1}^{s} \overline{\langle x^{a_i}, y^{b_i} \rangle} = \sum_{i=1}^{s} \left(x^{\sum_{i'=1}^{i-1} a_{i'}} y^{\sum_{i'=i+1}^{s} b_{i'}} \right) \overline{\langle x^{a_i}, y^{b_i} \rangle}. \tag{2.1}$$

In $\mathbb{N}^2$ we illustrate this product as the vertices $(a_1 + \cdots + a_i, b_{i+1} + \ldots + b_s) = (\mathrm{a}_i, \mathrm{b}_i)$, $0 \leq i \leq s$, and the diagonal lines between two consecutive vertices. Then

$\mathrm{conv}(\cup(\mathrm{a}_i, \mathrm{b}_i)+\mathbb{N}^2)$ constitutes the log-set of the product. If we keep only those vertices $(\mathrm{a}_j, \mathrm{b}_j)$ that satisfy the condition $\frac{a_i}{b_i} < \frac{a_{i+1}}{b_{i+1}}$, then the convex hull of $\cup(\mathrm{a}_i, \mathrm{b}_i) + \mathbb{N}^2$ will also constitute the desired log-set. Thus, the generators of the minimal monomial reduction are illustrated by such vertices $(\mathrm{a}_j, \mathrm{b}_j)$ that $\frac{\mathrm{a}_j - \mathrm{a}_{j-1}}{\mathrm{b}_{j-1} - \mathrm{b}_j} < \frac{\mathrm{a}_{j+1} - \mathrm{a}_j}{\mathrm{b}_j - \mathrm{b}_{j+1}}$.

Example 2.4. The ideal $\bar{I}$ in the previous example is a product of the blocks $\overline{\langle x^2, y^5\rangle}$, $\overline{\langle x^2, y^3\rangle}$, $\overline{\langle x^2, y^3\rangle}$, $\overline{\langle x^2, y\rangle}$. It is depicted by the vertices (12,0), (2,7), (4,4), (6,1) and (8,0), where (4,4) is omitted to give the least monomial reduction, since $\overline{\langle y^{12}, x^2y^7, x^6y, x^8\rangle} = \bar{I}$.

The reduction number of an integrally closed ideal is two (Theorem 5.1 [4]). Hence, $I_{lmr}\bar{I} = \bar{I}^2$.

Proposition 2.5. *Any power of a simple and* $\mathfrak{m}$*-primary integrally closed monomial ideal in* $k[[x, y]]$ *(or* $k[x, y]$*)) has the ideal* $\langle y^b, x^a\rangle$ *as its minimal reduction, where* b *denotes the highest power of* y *and* a *the highest power of* x *in the minimal set of generators.*

Proof. There are two types of such ideals [2].

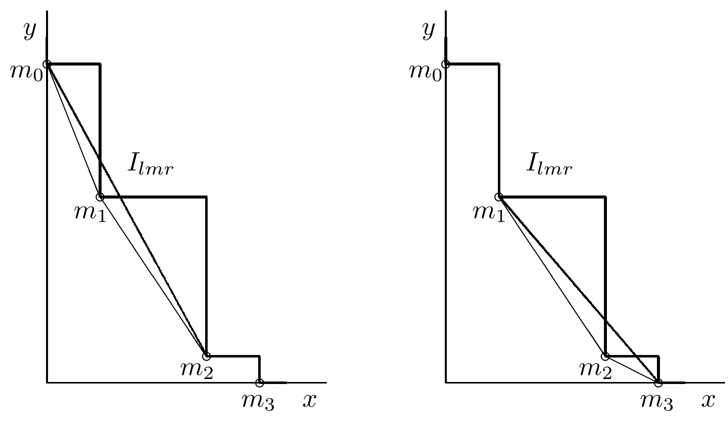
Let $I = \langle x^j y^{B_j}\rangle_{0\le j\le s}$ where $B_j = \lceil \frac{s-j}{s} B_0\rceil$. Theorem 3.8 in [2] states that $I^2 = \langle x^j y^{B_0+B_j}, x^{s+j}y^{B_j}\rangle$ and it is obvious that $I\langle y^{B_0}, x^s\rangle = I^2$.

Let $I = \langle x^{A_i} y^{r-i}\rangle_{0\le i\le r}$ where $A_i = \lceil i\frac{A_r}{r}\rceil$. Then $I^2 = \langle x^{A_i}y^{2r-i}, x^{A_r+A_i}y^{r-i}\rangle$ and the result follows similarly. $\square$

Assume that the integral closure of a monomial ideal is some power of a block. Then the least monomial reduction is generated by two monomials and is, of course, a minimal reduction itself.

Suppose that the least monomial reduction of an ideal is generated by more than two monomials. Then the generators satisfy a certain condition which we at first will demonstrate by an example.

Example 2.6. Let $I_{lmr} = \langle y^{12}, x^2y^7, x^6y, x^8\rangle = \langle m_j\rangle_{0\le j\le 3}$ as previously.



It is clearly seen that moving m_1 either vertically or horisontally it will intersect the line between m_0 and m_2.The same is valid for m_2 and the line between m_1 and m_3. The pictures correspond to certain relations between any three consecutive generators. We have

$$m_1^3 y \mid m_0^2 m_2 \text{ and } m_2^3 y \mid m_1 m_3^2. \tag{2.2}$$

In (2.2) we could as well have chosen x instead of y and get:

$$m_1^5 x \mid m_0^3 m_2^2 \text{ and } m_2^4 x \mid m_1 m_3^3. \tag{2.3}$$

Any I_{lmr} is constructed in such a way that there are relations similar to (2.2) between the generators. Let $I_{lmr} = \langle x^{A_j} y^{B_j} \rangle = \langle m_j \rangle_{0 \leq j \geq r}$. Let further $c_j = \mathrm{A}_{j+1} - \mathrm{A}_j$ and $l_j = \mathrm{A}_{j+1} - \mathrm{A}_{j-1}$. Then we can easily deduce that

$$\begin{cases} l_j \mathrm{A}_j & = c_j \mathrm{A}_{j-1} + (l_j - c_j)\mathrm{A}_{j+1} \\ l_j \mathrm{B}_j + 1 & \leq c_j \mathrm{B}_{j-1} + (l_j - c_j)\mathrm{B}_{j+1}, \quad \text{that is, } m_j^{l_j} y \mid m_{j-1}^{c_j} m_{j+1}^{l_j - c_j}. \end{cases} \tag{2.4}$$

Remark 2.7. The results that will follow will depend on the ring being local. Henceforth we will consider only the formal power series ring.

Often we can choose smaller l_j, compared to Example 2.6 where $c_1 = 4$ and $l_1 = 6$ to start with. In the local ring with the maximal ideal $\mathfrak{m} = \langle x, y \rangle$ the expression (1.4) says

$$m_{j-1}^{c_j} m_{j+1}^{l_j - c_j} \in m_j^{l_j} \mathfrak{m}. \tag{2.5}$$

By taking $l = \mathrm{lcm}(l_j)$, we can assume that all l_j are equal. The relation between three consecutive generators can be extended to three arbitrary generators. For example, for any $0 < j < r - 2$ we have:

$$m_{j-1}^{c_j l} (m_j^{c_{j+1}} m_{j+2}^{l - c_{j+1}})^{l - c_j} \in (m_{j-1}^{c_j} m_{j+1}^{l - c_j})^l \mathfrak{m} \subseteq m_j^{l^2} \mathfrak{m}.$$

The power products m_j are nonzerodivisors, hence

$$m_{j-1}^{c'} m_{j+2}^{l' - c'} \in m_j^{l'} \mathfrak{m}.$$

This small result is worth generalizing in a lemma.

Lemma 2.8. *Let $(R, \mathfrak{m})$ be a local integral domain and $I = \langle m_j \rangle_{0 \leq j \leq r}$ an ideal in R. Suppose that the generators fulfil (2.5). Then, for each triple of indices $i < i' < j$ there are positive integers c and l, $c < l$, such that*

$$m_i^c m_j^{l-c} \in m_{i'}^l \mathfrak{m}. \tag{2.6}$$

There is an alternative way to express (2.6). For any pair of indices i and j, $j - i \geq 2$, there are positive integers c and l, $c < l$, such that $m_i^c m_j^{l-c} \in I^l \mathfrak{m}$. Multiplying by a proper power of some of the two generators we can formulate the following result.

Proposition 2.9. *Let $I_{lmr} = \langle m_i \rangle$ be a least monomial reduction of some ideal I in $k[[x, y]]$. Assume that its generators are ordered by descending powers of y (or x). Then there is an integer l such that $m_i^l m_j^l \in I^{2l}\mathfrak{m}$ for any two indices i and j such that $j - i \geq 2$.*

In the next section we determine minimal reductions for a class of ideals in any local commutative ring and will show that the least monomial reductions we have defined belong to this class. In that way we will be able to determine a minimal reduction J of any monomial ideal I in $k[[x, y]]$, because $J \subseteq I_{lmr} \subseteq I \subseteq \bar{I}$ where I_{lmr} is integral over J. Hence, J is a minimal reduction of any ideal between J and $\bar{I}$.

3 Minimal reductions

Let $(R, \mathfrak{m})$ be a local ring and $I = \langle m_i \rangle_{0 \leq i \leq r}$ an ideal in it. Suppose that the generators are ordered in such way that they satisfy the condition

$$m_i^l m_j^l \in I^{2l}\mathfrak{m} \tag{3.1}$$

for some integer l if $j - i \geq 2$.

A reduction of an ideal, the generators of which satisfy (3.1), can be expressed in a quite convenient way. Before we prove our theorem we need two lemmas. The first one is found on p.147 in [6].

Lemma 3.1. *Let $J \subseteq I$ be ideals. Then J is a reduction of I if and only if $J + I\mathfrak{m}$ is a reduction of I.*

Proof. If $JI^l = I^{l+1}$, then $(J + I\mathfrak{m})I^l = JI^l + I^{l+1}\mathfrak{m} = I^{l+1}$.

If $(J + I\mathfrak{m})I^l = I^{l+1}$, then we use Nakayama's lemma on $\mathfrak{m}(I^{l+1}/JI^l) = (I^{l+1}\mathfrak{m} + JI^l)/JI^l = I^{l+1}/JI^l$ which gives us $I^{l+1}/JI^l = \bar{0}$ and, hence, $I^{l+1} = JI^l$. The proof is complete. □

Lemma 3.2. *Let $I = \langle m_i \rangle_{0 \leq i \leq r}$ be an ideal. Assume further that $m_i^l \in JI^{l-1} + I^l\mathfrak{m}$ for all i and some integer l. Then J is a reduction of I.*

Proof. Let $l' = (l-1)r$, then $I^{l'+1} = \langle \prod_{i=0}^r m_i^{l_i} \mid \sum_{i=0}^r l_i = l' + 1 \rangle$. For every generator (product) there is some index k such that $l_k \geq l$, according to the pigeon hole principle. Then $m_k^{l_k} \in (JI^{l-1} + I^l\mathfrak{m})I^{l_k - l}$ and, hence, $\prod_{i=0}^r m_i^{l_i} = m_k^{l_k}\left(\prod_{i \neq k} m_i^{l_i}\right) \in JI^{l'} + I^{l'+1}\mathfrak{m}$. Thus, $I^{l'+1} \subseteq JI^{l'} + I^{l'+1}\mathfrak{m}$ and we are done due to Lemma 3.1. □

Theorem 3.3. *Let $I = \langle m_i \rangle_{0 \leq i \leq r}$ be an ideal in $(R, \mathfrak{m})$. Assume that there is a partition $\{0, \ldots r\} = \cup_{0 \leq \alpha \leq s} S_\alpha$, where $s \leq r$, such that if $i, j \in S_\alpha$, $i \neq j$, then $m_i^l m_j^l \in I^{2l}\mathfrak{m}$ for some integer l. Let $J = \langle \sum_{i \in S_\alpha} m_i \rangle_{0 \leq \alpha \leq s}$, then J is a reduction of I.*

Proof. In the case when every $|S_\alpha| = 1$, the ideal $J = I$ is trivially a reduction.

Suppose that $|S_\alpha| \geq 2$ for some α. For that α define $p_\alpha = \sum_{i \in S_\alpha} m_i$ and fix a $k \in S_\alpha$. By assumption $m_k^l m_i^l \in I^{2l}\mathfrak{m}$ for all $k \neq i \in S_\alpha$. Let $l' = |S_\alpha| \cdot (l-1)$. Then for any t, $0 \leq t \leq l + l' - 1$ we have

$$\begin{aligned} m_k^{l+l'-t}(p_\alpha - m_k)^{t+1} + m_k^{l+l'-t-1}(p_\alpha - m_k)^{t+2} = \\ = m_k^{l+l'-t-1}(p_\alpha - m_k)^{t+1} p_\alpha \in I^{l+l'} J. \end{aligned} \tag{3.2}$$

Using this we can rewrite the zero element in the quotient ring $R/JI^{l+l'}$ as:

$$\begin{aligned} [m_k^{l+l'} p_\alpha] = [m_k^{l+l'+1} + m_k^{l+l'}(p_\alpha - m_k)] \overset{(1.8)}{=} [m_k^{l+l'+1} - m_k^{l+l'-1}(p_\alpha - m_k)^2] = \ldots \\ \ldots = [m_k^{l+l'+1} \pm m_k^l (p_\alpha - m_k)^{l'+1}] = [m_k^{l+l'+1} \pm m_k^l (\sum_{\substack{\sum l_i = \\ l'+1}} \beta_{\ldots} (\prod_{\substack{i \neq k, \\ i \in S_\alpha}} m_i^{l_j}))], \end{aligned} \tag{3.3}$$

where the $\beta_{\ldots}$'s are the multinomial coefficients. According to the pigeon hole principle there is some k' in each product $\prod m_i^{l_i}$ such that $l_{k'} \geq l$. For that k' we have

$$m_k^l (\prod_{\substack{i \neq k, \\ \sum l_i = l'+1}} m_i^{l_i}) \in I^{2l}\mathfrak{m} I^{l'+1-l} = I^{l+l'+1}\mathfrak{m}. \tag{3.4}$$

From (3.3) and (3.4) we can deduce that $m_k^{l+l'+1} \in JI^{l+l'} + I^{l+l'+1}\mathfrak{m}$. Since the index k was chosen arbitrarily it follows easily that there is an integer L such that $m_i^L \in JI^{L-1} + I^L\mathfrak{m}$ for all i. Lemma 3.2 completes the proof. $\square$

Example 3.4. Let $I = \langle x^3yz^2, x^2y^2z, xy^3z, y^4z^2 \rangle = \langle m_i \rangle_{0 \leq i \leq 3} \subset R = k[[x, y, z]]$. The partition of the indices is $\{0, 2\} \cup \{1, 3\}$, since $m_0 m_2 \in m_1^2 \mathfrak{m}$ and $m_1 m_3 \in m_2^2 \mathfrak{m}$. Applying Theorem 3.3 gives us a minimal reduction of I which is $\langle x^3yz^2 + xy^3z, x^2y^2z + y^4z^2 \rangle$.

Example 3.5. Let $I = \langle x^4, x^2y, y^4, y^2z, z^4, z^2x \rangle = \langle m_i \rangle_{0 \leq i \leq 5} \subset R = k[[x, y, z]]$. This is an $\mathfrak{m}$-primary ideal, so the number of the generators of a reduction of it must be at least three. We see that $(x^2y)^2z \,|\, (x^4)(y^2z)$ or, equivalently, $m_0 m_3 \in (x^2y^2)\mathfrak{m}$. By symmetry there are similar relations for the other generators. Hence, we get a partition $\{0, 3\} \cup \{2, 5\} \cup \{1, 4\}$ and a minimal reduction $\langle x^4 + y^2z, y^4 + z^2x, z^4 + x^2y \rangle$.

We return to the monomial ideals in $k[[x, y]]$. The minimal reductions of an $\mathfrak{m}$-primary ideal is generated by two elements in this ring.

Corollary 3.6. *Let $I = \langle m_j \rangle_{0 \leq j \leq s}$ be a monomial ideal in $k[[x, y]]$ and $I_{lmr} = \langle m_i \rangle_{0 \leq i \leq r}$ its monomial reduction where the generators are ordered in such way that the powers of y (or x) are descending. Then $J = \langle \sum_{\text{even } i} m_i, \sum_{\text{odd } i} m_i \rangle$ is a minimal reduction of I.*

Moreover, if J is a minimal reduction of any ideal between J and $\bar{I}$.

Proof. We have shown that the relation (3.1) is valid for all the generators of I_{lmr} except for any two consecutive. Since two consecutive generators must lie in different subsets of the partition, a split into even and odd indices is the only one. The rest follows from the theorem. $\square$

Example 3.7. Consider Example 2.2. The ideal $J = \langle y^{12} + x^6 y, x^2 y^7 + x^8 \rangle$ is a minimal reduction of all ideals lying between J and $\bar{I}$.

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