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The Poincaré series of the module of derivations of some monomial rings

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Abstract

Let R be a graded k-algebra and M be a finitely generated R-module. The Poincaré series $P_M^R(z)$ is the formal power series $\sum_i \dim_k \operatorname{Tor}_i^R(k, M)z^i$. We determine the Poincaré series of the module of derivations of some monomial rings.

MSC: 13D02; 13D07

1 Introduction

If R is a commutative k-algebra, with k a field, the *module of derivations*, $\operatorname{Der}_k(R) \subseteq \operatorname{Hom}_k(R, R)$ is the set $\{\rho \in \operatorname{Hom}_k(R, R) \mid \rho(ab) = a\rho(b) + \rho(a)b$ for every $a, b \in R\}$. This set has a natural R-module structure by multiplication from left by elements in R.

Let R be a graded k-algebra and M be a finitely generated R-module. The *Poincaré series* $P_M^R(z)$ is the formal power series $\sum_i \dim_k \operatorname{Tor}_i^R(k, M)z^i$.

Our object of study in this paper is the Poincaré series of the module of derivations of some monomial rings.

1.1 Description of the content

We now make a closer description of the paper. In Section 2 we state a theorem which represents the starting point of our paper. Moreover we give some results which allow us to calculate the Poincaré series of the module of derivations of monomial rings.

In Section 3 we use results of Section 2 in order to determine the Poincaré series of the module of derivations for some classes of example of monomial rings. In particular we determine the Poincaré series of the module of derivations of complete intersection rings (cf. Subsection 3.1), of the rings of the type $k[X_1,\dots,X_n]/(X_1,\dots,X_n)^l$ (cf. Subsection 3.2) and of some Stanley-Reisner rings (cf. Subsection 3.3).

2 Preliminaries

The starting point of our paper is the following theorem due to Brumatti and Simis in [2].

Theorem 2.1. *Let $I \subseteq S = k[X_1,\dots,X_n]$ be an ideal generated by monomials whose exponents are prime to the characteristic of k. Then*

$$\mathrm{Der}_k(S/I) = \oplus_{i=1}^n (I : (I : X_i)/I)\partial_i$$

where $\partial_i = \frac{\partial}{\partial X_i}$.

To calculate the Poincaré series of the module of derivations of monomial rings we use the following proposition.

Proposition 2.2. *Let R be a ring and let J be an ideal in R. Then $P_J^R(z) = (P_{R/J}^R(z) - 1)/z$.*

Proof. This follows easily by comparing the minimal free R-resolutions of J and R/J.

In [5], Levin introduces the idea of a large homomorphism of graded (or local) rings as a dual notion to small homomorphism of graded rings introduced in [1]. Namely, if A and B are graded rings and $f : A \longrightarrow B$ is a graded homomorphism which is surjective, then f is *large* if $f_* : \mathrm{Tor}^A(k,k) \longrightarrow \mathrm{Tor}^B(k,k)$ is surjective.

Proposition 2.3. *Let R be a monomial ring and let $x_1,\dots,x_n$ be variables in R. Then the map $f : R \longrightarrow R/(x_1,\dots,x_n)$ is large.*

Proof. It is enough to prove that the map $f : R \longrightarrow R/(x)$, with x a variable in R, is large. Let us consider the following minimal free R-resolution of k

$$0 \longleftarrow k \longleftarrow R \longleftarrow R^{b_1} \longleftarrow \cdots .$$

This resolution is a multigraded resolution. If we kill everything with degree grater than zero in the variable x, we get the following minimal free $R/(x)$-resolution of k

$$0 \longleftarrow k \longleftarrow R/(x) \longleftarrow [R/(x)]^{b_1'} \longleftarrow \cdots .$$

Since all the vertical maps in

$$\begin{array}{ccccccccc} 0 & \longleftarrow & k & \longleftarrow & R & \longleftarrow & R^{b_1} & \longleftarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & k & \longleftarrow & R/(x) & \longleftarrow & [R/(x)]^{b_1'} & \longleftarrow & \cdots \end{array}$$

are surjective, then the homomorphism $f_* : \mathrm{Tor}^R(k,k) \longrightarrow \mathrm{Tor}^{R/(x)}(k,k)$ is surjective.

Corollary 2.4. *Let R and x be as in Proposition 2.3, then*

$$P^R_{R/(x)}(z) = \frac{P^R_k(z)}{P^{R/(x)}_k(z)}.$$

Proof. This follows by Proposition 2.3 and [5, Theorem 1.1].

3 Examples

In this section we use results in the Preliminares in order to determine the Poincaré series of the module of derivations of some classes of example of monomial rings.

3.1 The complete intersection case

Let R be a complete intersection monomial ring. Then we can suppose

$$R = \frac{k[X_1, \ldots, X_n]}{(X_1^{n_1} \cdots X_{m_1}^{n_{m_1}}, X_{m_1+1}^{n_{m_1+1}} \cdots X_{m_2}^{n_{m_2}}, \ldots, X_{m_{r-1}+1}^{n_{m_{r-1}+1}} \cdots X_{m_r}^{n_{m_r}})}$$

with $m_r \leq n$.

Let us suppose that $n_1, \ldots, n_{m_r}$ are prime to the characteristic of k. Then, by Theorem 2.1, we get $M = x_1\partial_1 \oplus \cdots \oplus x_{m_r}\partial_{m_r} \oplus R\partial_{m_r+1} \oplus \cdots \oplus R\partial_n$.

By [7, Theorem 6], Corollary 2.4 and Proposition 2.2, we have, for every $i = 1, \ldots, m_r$, that $P^R_{(x_i)}(z) = 1/1 - z$.

Finally, since $P^R_R(z) = 1$, we get

$$P^R_M(z) = \frac{n + (m_r - n)z}{1 - z}.$$

3.2 The case of $k[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^l$

Let $R = k[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^l = k[X_1, \ldots, X_n]/\mathfrak{m}^l$ and let us suppose that l and $\mathrm{char}(k)$ are relative primes.

Since $0 : (0 : X_i) = 0 : \mathfrak{m}^{l-1} = \mathfrak{m}$, then, by Theorem 2.1, we get $M = \mathfrak{m}\partial_1 \oplus \cdots \oplus \mathfrak{m}\partial_n$.

By Corollary 2.4 and [4, Pag 748], we get

$$P^R_{R/\mathfrak{m}}(z) = P^R_k(z) = \frac{(1+z)^n}{1 - \sum_{i=1}^n \binom{i+l-2}{l-1}\binom{n+l-1}{i+l-1} z^{i+1}}.$$

Finally, using Proposition 2.2, we get

$$P^R_M(z) = n\frac{\frac{(1+z)^n}{1 - \sum_{i=1}^n \binom{i+l-2}{l-1}\binom{n+l-1}{i+l-1} z^{i+1}} - 1}{z}.$$

3.3 The case of some Stanley-Reisner rings

In this subsection we determine the Poincaré series of the module of derivations of some Stanley-Reisner rings.

A *(finite) simplicial complex* consists of a finite set V of vertices and a collection Δ of subsets of V called *faces* or *simplices* such that
(i) If $v \in V$, then $\{v\} \in \Delta$.
(ii) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

Let Δ be a simplicial complex and $F \in \Delta$, then the *dimensions* of F and Δ are defined by $\dim(F) = |F| - 1$ and $\dim(\Delta) = \sup\{\dim(F) \mid F \in \Delta\}$ respectively. A face of dimension q is sometimes refer to a q-face.

Let $S = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k and let Δ be a simplicial complex with vertex set $V = \{X_1, \ldots, X_n\}$. The *Stanley-Reisner ring* $k[\Delta]$ is defined as the quotient ring S/I, where

$$I = (\{X_{i_1} \cdots X_{i_r} \mid i_1 < \cdots < i_r,\ \{X_{i_1} \cdots X_{i_r}\} \notin \Delta\}).$$

For a general reference to properties of simplicial complexes and of a Stanley-Reisner ring, see [8].

To calculate Hilbert series of Stanley-Reisner rings, the following lemma is helpful.

Lemma 3.1. *[6, Theorem 2.1.4] Let f_i be the number of i-dimensional faces of a $(d-1)$-dimensional simplicial complex Δ. Then $H_{k[\Delta]}(z) = \sum_{i=-1}^{d-1} f_i z^{i+1}/(1-z)^{i+1}$.*

3.3.1 The graphs case

We first start to determine the Poincaré series of the module of derivations of some Stanley-Reisner rings R of simplicial complexes of dimension one. To this purpose we use Theorem 2.1 and Corollary 3.3 to give a method to determine the Poincaré series of $\mathrm{Der}_k k[\Delta]$ for some Stanley-Reisner ring with relations of degree 2. Anyway, we note that this method works equally well (if char $(k) \neq 2$) for all monomial rings with relations of degree 2.

Let $R = k[X_1, \ldots, X_n]/I$ where I is generated by monomials of degree 2 and let $\mathfrak{b}$ be an ideal generated by a subset of $\{X_1, \ldots, X_n\}$.

In [3, Sect. 1] the authors introduce the so called Koszul filtration. It follows from [3, Proposition 1.2] that if R and $\mathfrak{b}$ are as above, then R has a Koszul filtration and $\mathfrak{b}$ has a linear free R-resolution.

The following theorem is probably well known, but we prove it for the convenience of the reader.

Theorem 3.2. *Let R and $\mathfrak{b}$ be as above. Then $H_R(z)P^R_{\frac{R}{\mathfrak{b}}}(-z) = H_{\frac{R}{\mathfrak{b}}}(-z)$.*

Proof. Since R has a Koszul filtration, we have the following free linear R-resolution of $R/\mathfrak{b}$

$$\cdots \longrightarrow R^{b_3}[-3] \longrightarrow R^{b_2}[-2] \longrightarrow R^{b_1}[-1] \longrightarrow R^{b_0} = R \longrightarrow R/\mathfrak{b} \longrightarrow 0.$$

Let $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ and $R/\mathfrak{b} = k \oplus [R/\mathfrak{b}]_1 \oplus [R/\mathfrak{b}]_2 \oplus \cdots$, then we have the following graded version of the resolution above

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
0 & \longleftarrow & [R/\mathfrak{b}]_3 & \longleftarrow & R_3 & \longleftarrow & R_2^{b_1} & \longleftarrow & R_1^{b_2} & \longleftarrow & k^{b_3} \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \longleftarrow & [R/\mathfrak{b}]_2 & \longleftarrow & R_2 & \longleftarrow & R_1^{b_1} & \longleftarrow & k^{b_2} & & \\
 & & \oplus & & \oplus & & \oplus & & & & \\
0 & \longleftarrow & [R/\mathfrak{b}]_1 & \longleftarrow & R_1 & \longleftarrow & k^{b_1} & & & & \\
 & & \oplus & & \oplus & & & & & & \\
0 & \longleftarrow & k & \longleftarrow & k & & & & & &
\end{array}
$$

Hence we get the following exact sequence of vector spaces (with $m > 0$)

$$0 \longrightarrow k^{b_m} \longrightarrow R_1^{b_{m-1}} \longrightarrow R_2^{b_{m-2}} \longrightarrow \ldots \longrightarrow R_{m-1}^{b_1} \longrightarrow R_m \longrightarrow [R/\mathfrak{b}]_m \longrightarrow 0.$$

Let $\dim_k R_i = h_i$ and let $\dim_k [R/\mathfrak{b}]_i = r_i$ and in particular $h_0 = r_0 = 1$. Then, for every $i \geq 0$, we have $h_0 b_i - h_1 b_{i-1} + \cdots + (-1)^i h_i b_0 = (-1)^i r_i$, hence $(h_0 + h_1 z + h_2 z^2 + \cdots)(b_0 - b_1 z + b_2 z^2 - \cdots) = (r_0 - r_1 z + r_2 z^2 - \cdots)$.

Corollary 3.3. $P_{\mathfrak{b}}^R(z) = \left(H_{\frac{R}{\mathfrak{b}}}(z)/H_R(-z) - 1 \right)/z.$

Proof. This follows by Theorem 3.2 since $P_{\mathfrak{b}}^R(z) = (P_{\frac{R}{\mathfrak{b}}}^R(z) - 1)/z$.

From now on, if $R = k[X_1, \ldots, X_n]/I$, then $\widetilde{R}$ will denote the ring $k[X_1, \ldots, X_n]/(I, X_1^2, \ldots, X_n^2)$.

Let us start to determine the Poincaré series of the module of derivations of Stanley-Reisner rings for a cycle.

Let Δ be a cycle with vertex set $V = \{X_1, \ldots, X_n\}$, $n \geq 3$. If $n = 3$, then $R = k[\Delta] = k[X_1, X_2, X_3]/(X_1 X_2 X_3)$ is a complete intersection and $P_M^R(z) = 4/1 - z$ (cf. Subsection 3.1).

Let us consider $n \geq 4$. Then $R = k[\Delta] = k[X_1, \ldots, X_n]/(X_i X_j :\mid i - j \mid \geq 2 \pmod{n}) = k[X_1, \ldots, X_n]/I$.

By Theorem 2.1, we get $M = x_1 \partial_1 \oplus \cdots \oplus x_n \partial_n$.

Then, $H_{\widetilde{R}}(z) = 1 + nz + (n-1)z^2$ and, by Lemma 3.1, we get $H_R(z) = H_{\widetilde{R}}(z/1-z) = 1+(n-2)z/(1-z)^2$. Since $H_{\widetilde{R}/(x_i)}(z) = 1+(n-1)z+(n-3)z^2$, for every $i = 1,\dots,n$, we get $H_{R/(x_i)} = 1 + (n-3)z - z^2/(1-z)^2$.

Hence, by Corollary 3.3, for every $i = 1,\dots,n$, we have

$$P^R_{(x_i)}(z) = \frac{(2n-1) - 3z - (2n-7)z^2 - z^3}{(1-z)^2(1-(n-2)z)}.$$

Finally

$$P^R_M(z) = n\frac{2n-1-3z-(2n-7)z^2-z^3}{(1-z)^2(1-(n-2)z)}.$$

Let now Δ be a complete bipartite graph $K_{m,n}$ with vertex set $V = \{X_1,\dots,X_{m+n}\}$.

The case $k_{n-1,1}$ is treated below as a path if $n = 2,3$ and as a star graph if $n \geq 4$. Moreover, since $K_{2,2}$ is a cycle, we can suppose $m \geq 3$ and $n \geq 2$. Then $R = k[\Delta] = k[X_1,\dots,X_{m+n}]/(X_iX_j \mid i \neq j; i,j \in \{1,\dots,n\},$ or $i,j \in \{m+1,\dots,m+n\}) = k[X_1,\dots,X_{m+n}]/I$.

By Theorem 2.1, we get $M = x_1\partial_1 \oplus \cdots \oplus x_n\partial_n$.

Using the same method as for the cycle case we get that $H_R(z) = 1 + (m+n-2)z + (mn-m-n+1)z^2/(1-z)^2$ and, for every $i = 1,\dots,m$, $H_{R/(x_i)} = 1 + (m+n-3)z + (mn+2-m-2n)z^2/(1-z)^2$ and, for every $i = m+1,\dots,n$, $H_{R/(x_i)} = 1+(m+n-3)z+(mn+2-2m-n)z^2/(1-z)^2$.

Hence for every $i = 1,\dots,m$, we get

$$P^R_{(x_i)}(z) = \frac{2m+2n-1+(-n-1)z-(4mn-2m-4n+1)z^2+(1-n)z^3}{(1-z)^2(1-(m+n-2)z+(mn-m-n+1)z^2)}$$

and for every $i = m+1,\dots,m+n$ we get

$$P^R_{(x_i)}(z) = \frac{2m+2n-1+(-m-1)z-(4mn-4m-2n+1)z^2+(1-m)z^3}{(1-z)^2(1-(m+n-2)z+(mn-m-n+1)z^2)}.$$

Finally

$$P^R_M(z) = \frac{a+bz-cz^2+dz^3}{(1-z)^2(1-(m+n-2)z+(mn-m-n+1)z^2)}.$$

with $a = 2m^2+2n^2+4mn-m-n$, $b = -2mn-m-n$, $c = -4m^2n - 4mn^2+2m^2+2n^2+8mn-m-n$ and $d = m+n-2mn$.

Let us now consider the case of a tree graph.

Let v be a vertex of a graph Δ. The degree of v, $\deg(v)$, is the number of edges at v. We denote by $N_\Delta(v)$ the set of neighbours of a vertex v.

Now let Δ be a tree. If $\deg(v) = 1$, then v is called *leaf*. We call v an *almost leaf* if there is a leaf w in $N_\Delta(v)$. Finally we call v an *inner point* if v is not a leaf and no leaves are in $N_\Delta(v)$.

Let $R = k[X_1, \ldots, X_n]/I = k[\Delta]$ be the Stanley-Reisner ring of a tree Δ with vertex set $V = \{X_1, \ldots, X_n\}$. It is easy to see that $0 : (0 : X_i) = X_i$ whenever X_i is a leaf or an inner point and $0 : (0 : X_i) = (X_i, X_{i_1}, \ldots, X_{i_r})$ whenever X_i is an almost leaf and $X_{i_1}, \ldots, X_{i_r}$ are the leaves in $N_\Delta(X_i)$.

Let Δ and R be as above. Then $H_R(z) = 1 + (n-2)z/(1-z)^2$.

If X_i is a leaf, then $H_{R/(x_i)} = 1 + (n-3)z/(1-z)^2$ and if X_i, hence

$$P^R_{(x_i)}(z) = \frac{2n - 1 - 2z + (2n-5)z^2}{(1-z)^2(1-(n-2)z)}.$$

If X_i is an inner point with $N_\Delta(X_i) = \{X_{i_1}, \ldots, X_{i_r}\}$, then $H_{R/(x_i)} = 1 + (n-3)z + (1-r)z^2/(1-z)^2$. Hence

$$P^R_{(x_i)}(z) = \frac{2n - 1 + (1-r)z + (2n-2r-3)z^2 + (1-r)z^3}{(1-z)^2(1-(n-2)z)}.$$

Finally if X_i is an almost leaf with $N_\Delta(X_i) = \{X_{i_1}, \ldots, X_{i_r}, X_{j_1}, \ldots, X_{j_s}\}$, where $X_{i_1}, \ldots, X_{i_r}$ are the leaves in $N_\Delta(X_i)$, then $H_{R/(x_1, x_{i_1}, \ldots, x_{i_r})} = 1 + (n - r - 3)z + (1-s)z^2/(1-z)^2$. Hence

$$P_{(x_1, x_{i_1}, \ldots, x_{i_r})}(z) = \frac{2n - r - 1 + (-1-s-2r)z + (2n-s-r-3)z^2 + (1-s)z^3}{(1-z)^2(1-(n-2)z)}.$$

Let us now consider some special classes of trees.

Let Δ be a path with vertex set $V = \{X_1, \ldots, X_n\}$.

If $n = 2$, then $R = k[X_1, X_2]$ and $M \simeq R^2$. Hence $P^R_M(z) = 2$.

Let us suppose $n \geq 3$. Then $R = k[\Delta] = k[X_1, \ldots, X_n]/(X_iX_j \mid i \in \{1, \ldots, n-2\}, j > i+1) = k[X_1, \ldots, X_n]/I$.

By what is written above, we get

$$M = x_1\partial_1 \oplus (x_1, x_2)\partial_2 \oplus x_3\partial_3 \oplus \cdots \oplus x_{n-2}\partial_{n-2} \oplus (x_{n-1}, x_n)\partial_{n-1} \oplus x_n\partial_n.$$

Moreover

$$P^R_{(x_1)}(z) = P^R_{(x_n)}(z) = \frac{2n - 1 - 2z + (2n-5)z^2}{(1-z)^2(1-(n-2)z)},$$

$$P^R_{(x_1,x_2)}(z) = P^R_{(x_{n-1},x_{n-2})}(z) = \frac{2n - 2 - 3z + (2n-4)z^2 + z^3}{(1-z)^2(1-(n-2)z)}.$$

and (if $n \geq 5$) for $i = 3, \ldots, n-2$, we get

$$P^R_{(x_i)}(z) = \frac{2n - 1 - 3z + (2n-7)z^2 - z^3}{(1-z)^2(1-(n-2)z)}.$$

Hence

$$P^R_M(z) = \frac{14 - 7z + 4z^2 + z^3}{(1-z)^3}$$

if $n = 3$,

$$P_M^R(z) = \frac{26 - 10z + 14z^2 + 2z^3}{(1-z)^2(1-2z)}$$

if $n = 4$ and

$$P_M^R(z) = \frac{2n^2 - n - 2 + (3n-16)z + (2n^2 - 7n - 1)z^2 + (4-n)z^3}{(1-z)^2(1-(n-2)z)}$$

if $n \geq 5$.

Let now Δ be a star graph with vertex set $V = \{X_1, \dots, X_n\}$, $n \geq 4$, and with center vertex X_n. Then $R = k[\Delta] = k[X_1, \dots, X_n]/(X_i X_j \mid i \in \{1, \dots, n-2\}, n > j > i) = k[X_1, \dots, X_n]/I$.

Moreover $M = x_1\partial_1 \oplus \cdots \oplus x_{n-1}\partial_{n-1} \oplus (x_1, \dots, x_n)\partial_n$ and, for every $i = 1, \dots, n-1$, we have

$$P_{(x_i)}^R(z) = \frac{2n - 1 - 2z + (2n-5)z^2}{(1-z)^2(1-(n-2)z)}.$$

and

$$P_{(x_1,\dots,x_n)}^R(z) = \frac{n+z}{1-(n-2)z}.$$

Finally

$$P_M^R(z) = \frac{2n^2 - 2n + 1 + (3-4n)z + (2n^2 - 6n + 3)z^2 + z^3}{(1-z)^2(1-(n-2)z)}.$$

By what is written above, we can find the Poincaré series of the module of derivations of the Stanley-Reisner ring of any arbitrary binary graph. Anyway in the case of a complete binary graph we give a general formula for the Poincaré series.

Let Δ be a complete binary graph with vertex set $V = \{X_1, \dots, X_n\}$ and t levels (hence $n = 2^{t+1} - 1$).

If $t = 1$, then Δ is a path. So we can suppose $t \geq 2$. Then $R = k[\Delta] = k[X_1, \dots, X_n]/(X_i X_j \mid i < j, j \neq 2i, 2i+1)$.

By what is written above, we get

$$M = x_1\partial_1 \oplus \cdots \oplus x_{2^{t-1}-1}\partial_{2^{t-1}-1} \oplus (x_{2^{t-1}}, x_{2^t}, x_{2^t+1})\partial_{2^{t-1}} \oplus$$

$$\oplus (x_{2^{t-1}+1}, x_{2^t+2}, x_{2^t+3})\partial_{2^t+1} \oplus \cdots \oplus$$

$$\oplus (x_{2^t-1}, x_{2^{t+1}-2}, x_{2^{t+1}-1})\partial_{2^t-1} \oplus x_{2^t}\partial_{2^t} \oplus \cdots \oplus x_{2^{t+1}-1}\partial_{2^{t+1}-1}.$$

Since there are $2^t = (n+1)/2$ leaves, $2^{t-1} = (n+1)/4$ almost leaves and $2^{t-1} - 1 = (n-3)/4$ inner points, we get

$$P_M^R(z) = \frac{8n^2 - 6n - 2 + (-12n - 4)z + (8n^2 - 25n + 11)z^2 + (6-2n)z^3}{4(1-z)^2(1-(n-2)z)}.$$

3.3.2 The case of skeletons of simplex

The q-*skeleton* of a simplicial complex Δ is the simplicial complex Δ^q consisting of all p-faces of Δ with $p \leq q$.

A simplicial complex Δ with vertex set V and with $|V| = m$ is called *simplex* if $\dim \Delta = m - 1$.

In this subsection we determine the the Poincaré series of the module of derivations for a Stanley-Reisner ring R of skeletons of simplex. We can even think of R as the factor ring of the polynomial ring modulo all squarefree monomials of a certain degree.

Let Δ_{n-1}^q be the q-dimensional skeleton of a $(n-1)$-dimensional simplex Δ_{n-1}.

If $q = n - 1$, then $R = k[X_1, \dots, X_n]$, $M \simeq R^n$ and $P_M^R(z) = n$.

Let us suppose hence that $q < n - 1$.

Then $R = k[\Delta] = k[X_1, \dots, X_n]/(X_{m_1} X_{m_2} \cdots X_{m_{q+2}} | m_1 < m_2 < \cdots < m_{q+2})$.

By Theorem 2.1, we get $M = x_1\partial_1 \oplus \cdots \oplus x_n\partial_n$.

By [8, Proposition 5.3.14], we have that R is Cohen-Macaulay. Using Lemma 3.1, we have that $H_R(z) = \sum_{i=0}^n \binom{n}{i} z^i/(1-z)^i$.

Let us now suppose that k is infinite. Then there exists a regular sequence of linear elements of length $q+1$, $\{a_1, \dots, a_{q+1}\}$.

Let us denote $R/(a_1, \dots, a_{q+1})$ by R'. Then the Hilbert series

$$H_{R'}(z) = (1-z)^{q+1} H_R(z) =$$

$$= 1 + \binom{n-(q+1)}{1} z + \binom{n-(q+1)+1}{2} z^2 + \cdots + \binom{n+q}{q+1} z^{q+1} = H_{\overline{R}}(z)$$

where $\overline{R} = k[Y_1, \dots, Y_{n-(q+1)}]/(Y_1, \dots, Y_{n-(q+1)})^{q+2}$. Hence $R' \simeq \overline{R}$.

Using the result in Subsection 3.2, we get

$$P_k^R(z) = (1+z)^{q+1} P_k^{R'}(z) = \frac{(1+z)^n}{1 - \sum_{i=1}^{n-(q+1)} \binom{i+q}{q+1}\binom{n+1}{i+q+1} z^{i+1}}.$$

Let x_i, with $i \in \{1, \dots, n\}$.

Since $q < n-1$, then $R/(x_i)$ is the Stanley-Reisner ring of Δ_q^{n-2}. Using the same argument as above, we get

$$P_k^{R/(x_i)}(z) = \frac{(1+z)^{n-1}}{1 - \sum_{i=1}^{n-(q+2)} \binom{i+q}{q+1}\binom{n}{i+q+1} z^{i+1}}.$$

Finally, using Corollary 2.4 and Proposition 2.2, we get

$$P_M^R(z) = n \frac{\frac{(1+z)(1 - \sum_{i=1}^{n-(q+2)} \binom{i+q}{q+1}\binom{n}{i+q+1} z^{i+1})}{1 - \sum_{i=1}^{n-(q+1)} \binom{i+q}{q+1}\binom{n+1}{i+q+1} z^{i+1}} - 1}{z}.$$

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