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# A classical tableau calculus

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July 28, 2003

**Abstract**

We refine Gentzen's classical sequent calculus into a classical tableau calculus within the framework of Martin-Löf's type theory. This calculus has essentially the same inference figures as Gentzen's classical sequent calculus, but there are two important departures from the latter: there are no structural inference figures except cut, and categories in the sense of Martin-Löf's type theory replace sequents.

The classical tableau calculus admits a normalization algorithm similar to the algorithm implicit in Gentzen's Hauptsatz, however confluent and strongly normalizing. The proof of the latter conforms with Gandy's technique, which in turn relies on the algorithm being weakly normalizing, here proved along the same line of argument as Gentzen's Hauptsatz.

## 1 Introduction

Gentzen's classical sequent calculus is in this paper refined into a classical tableau calculus without structural inference figures except cut, and a normalization algorithm similar to the algorithm implicit in Gentzen's Hauptsatz is developed. The normalization algorithm is confluent and strongly normalizing as opposed to the latter.

The classical tableau calculus and the normalization algorithm grew from an effort to extend the double negation interpretation to act on derivations as well as formulas, where they are used to level the ground. This effort has met with success, but is presented elsewhere [1], where also the rationale for

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the normalization algorithm can be found. This paper dwell on more formal matter.

The mathematical framework of this paper is Martin-Löf's type theory, and the basic reference work on this topic is [4], but see also [6]. It is here principally used to explain sequents, the identity of derivations, and signs (Sect. 2), the latter in the sense of classical tableau calculus, but also to speak about the classical connectives and quantifiers.

We shall designate the classical connectives and quantifiers by indeterminate constants (conjunction &, disjunction $\vee$, implication $\supset$, negation $\sim$, universal quantification $\forall$, and existential quantification $\exists$) of the appropriate types. This makes it possible for us to speak about the classical connectives and quantifiers without having to explain their meaning.

## 2 Sequents

We shall use Martin-Löf's type theory to interpret sequents, in intuitionistic logic and then classical predicate logic, as categories in the sense of type theory. Lower-case Greek letters will designate types and upper-case Latin letters will designate sets and propositions. The indeterminate proposition $\Psi$ of Section 2.2 is an exception to this rule.

In type theory, to know that something is a category is to know what its objects are and under what conditions two of its objects are equal. We want to speak about categories of old-fashioned functions,

$$\beta(x_1, ..., x_m)\ (x_1{:}\alpha_1, ..., x_m{:}\alpha_m(x_1, ..., x_{m-1})),$$

as metamathematical objects, or what amounts to the same thing, reflect them as objects of some other category. Anyhow, we have to explain under what conditions two categories of old-fashioned functions are equal: To know that two categories categories of old-fashioned functions, $\alpha(...)\ (...)$ and $\beta(...)\ (...)$, are equal is to know that an object of $\alpha(...)\ (...)$ is also an object of $\beta(...)\ (...)$ and, moreover, equal objects of $\alpha(...)\ (...)$ are also equal objects of $\beta(...)\ (...)$, and vice versa.

Contrary to the usual practice, we will for categories of old-fashioned functions admit expressions where each assumption may occur more than once, e.g. expressions like $\beta(x)\ (x{:}\alpha, x{:}\alpha)$, on the understanding that they are equal to their counterparts where each assumption occur only once.

## 2.1 Sequents in intuitionistic logic

Sequents and categories of old-fashioned functions have similar if not identical roles within type theory. We shall consider them to be synonymous concepts and thus identify each sequent of the form

$$x_1{:}\alpha_1, ..., x_m{:}\alpha_m(x_1, ..., x_{m-1}) \Rightarrow \beta(x_1, ..., x_m)$$

with the corresponding category

$$\beta(x_1, ..., x_m)\ (x_1{:}\alpha_1, ..., x_m{:}\alpha_m(x_1, ..., x_{m-1})).$$

We shall refer to categories of old-fashioned functions simply as categories. What we mean will always by clear from the context.

We shall from now on restrict our attention to predicate logic and thus to sequents of the form

$$\Delta, x_1{:}\alpha_1(d_1, ..., d_k), ..., x_m{:}\alpha_m(d_1, ..., d_k) \Rightarrow \beta(d_1, ..., d_k),$$

where $\Delta$ is some context $d_1{:}\mathsf{Element}(D), ..., d_k{:}\mathsf{Element}(D)$ of variables over the set $D$ of individuals. We shall suppress the context $\Delta$ as well as the variables over $D$ and write these sequents simply as

$$x_1{:}\alpha_1, ..., x_m{:}\alpha_m \Rightarrow \beta.$$

## 2.2 Sequents in classical logic

We adopt an indeterminate proposition $\Psi$ (from Greek $\psi\epsilon\acute{\upsilon}\delta o\varsigma$) and define the two signs of classical tableau calculus by

$$\begin{cases} \mathsf{T}A &= \mathsf{Proof}(A), \\ \mathsf{F}A &= (\mathsf{T}(A))\mathsf{T}(\Psi). \end{cases}$$

We can then translate each sequent $A_1, ..., A_m \rightarrow B_1, ..., B_n$ of classical predicate logic into a *tableau sequent*

$$x_1{:}\mathsf{T}A_1, ..., x_m{:}\mathsf{T}A_m, y_1{:}\mathsf{F}B_1, ..., y_m{:}\mathsf{F}B_n \Rightarrow \mathsf{T}\Psi$$

with distinct variables $x_1, ..., x_m, y_1, ..., y_n$ and then interpret the classical consequence relation accordingly. We will abbreviate tableau sequents by expressions like

$$\overset{x_1}{\mathsf{T}A_1}, ..., \overset{x_m}{\mathsf{T}A_m}, \overset{y_1}{\mathsf{F}B_1}, ..., \overset{y_m}{\mathsf{F}B_n}\ .$$

The two signs are closely related to the intuitionistic concepts of truth and falsity. These concepts are examined in great detail by Martin-Löf in [5], where he, among other things, explains why to judge a proposition $A$ true carries the same meaning as to judge the corresponding type $\mathsf{Proof}(A)$ inhabited, and why to judge a proposition $A$ false carries the same meaning as to judge the corresponding type $(\mathsf{Proof}(A))\mathsf{Proof}(\bot)$ inhabited. These two forms of judgement can be generalized to the corresponding forms of judgement made under assumptions.

# 3 The calculus C1

We shall refine Gentzen's classical sequent calculus [8] into a classical tableau calculus within the framework of Martin-Löf's type theory. This calculus has essentially the same inference figures as Gentzen's classical sequent calculus, but there are two important departures from the latter: there are no structural inference figures except cut, and tableau sequents replace sequents.

The inference figures of the classical tableau calculus are displayed in the table at page 5. They are organized into two columns and we call them left and right inference figures respectively. We furthermore call those inference figures neither axiom nor cut by the name of logical inference figures.

## 3.1 Restrictions on contexts and variables

An assumption discharged by an inference must not occur in the corresponding premiss context, and each assumption that occurs in a premiss context of an inference must also occur in the corresponding conclusion context, e.g. to be allowed to make an inference

$$\frac{\overset{u}{\mathsf{T}}A,\Gamma_1 \quad \overset{x}{\mathsf{F}}A,\Gamma_2}{\Gamma_3}\,{\scriptstyle cut\ u,x}, \tag{1}$$

the assumption $u{:}\mathsf{T}A$ must not occur in the context $\Gamma_1$, the assumption $x{:}\mathsf{F}A$ must not occur in the context $\Gamma_2$, and each assumption that occurs in either $\Gamma_1$ or $\Gamma_2$ must also occur in $\Gamma_3$.

For quantifiers we have the well known restrictions on variables. The variable bound by the quantifier in a right $\forall$-inference or a left $\exists$-inference must not occur free in any formula in the corresponding premiss context,

| | | | |
|---|---|---|---|
| *axiom* | $\dfrac{}{\overset{u}{\mathsf{T}}A, \overset{x}{\mathsf{F}}A, \Gamma}\ {\scriptstyle u,x}$ | *cut* | $\dfrac{\overset{u}{\mathsf{T}}A, \Gamma_1 \quad \overset{x}{\mathsf{F}}A, \Gamma_2}{\Gamma_3}\ {\scriptstyle u,x}$ |
| $\mathsf{T}\&$ | $\dfrac{\overset{u}{\mathsf{T}}A, \Gamma_1}{\overset{w}{\mathsf{T}}A\&B, \Gamma_2}\ {\scriptstyle 1;u;w}$ <br> $\dfrac{\overset{v}{\mathsf{T}}B, \Gamma_1}{\overset{w}{\mathsf{T}}A\&B, \Gamma_2}\ {\scriptstyle 2;v;w}$ | $\mathsf{F}\&$ | $\dfrac{\overset{x}{\mathsf{F}}A, \Gamma_1 \quad \overset{y}{\mathsf{F}}B, \Gamma_1}{\overset{z}{\mathsf{F}}A\&B, \Gamma_2}\ {\scriptstyle x,y;z}$ |
| $\mathsf{T}\vee$ | $\dfrac{\overset{u}{\mathsf{T}}A, \Gamma_1 \quad \overset{v}{\mathsf{T}}B, \Gamma_1}{\overset{w}{\mathsf{T}}A\vee B, \Gamma_2}\ {\scriptstyle u,v;w}$ | $\mathsf{F}\vee$ | $\dfrac{\overset{x}{\mathsf{F}}A, \Gamma_1}{\overset{z}{\mathsf{F}}A\vee B, \Gamma_2}\ {\scriptstyle 1;x;z}$ <br> $\dfrac{\overset{y}{\mathsf{F}}B, \Gamma_1}{\overset{z}{\mathsf{F}}A\vee B, \Gamma_2}\ {\scriptstyle 2;y;z}$ |
| $\mathsf{T}\supset$ | $\dfrac{\overset{x}{\mathsf{F}}A, \Gamma_1 \quad \overset{v}{\mathsf{T}}B, \Gamma_1}{\overset{w}{\mathsf{T}}A\supset B, \Gamma_2}\ {\scriptstyle x,v;w}$ | $\mathsf{F}\supset$ | $\dfrac{\overset{u}{\mathsf{T}}A, \overset{y}{\mathsf{F}}B, \Gamma_1}{\overset{z}{\mathsf{F}}A\supset B, \Gamma_2}\ {\scriptstyle u,y;z}$ |
| $\mathsf{T}\sim$ | $\dfrac{\overset{x}{\mathsf{F}}A, \Gamma_1}{\overset{w}{\mathsf{T}}\sim A, \Gamma_2}\ {\scriptstyle x;w}$ | $\mathsf{F}\sim$ | $\dfrac{\overset{u}{\mathsf{T}}A, \Gamma_1}{\overset{z}{\mathsf{F}}\sim A, \Gamma_2}\ {\scriptstyle u;z}$ |
| $\mathsf{T}\forall$ | $\dfrac{\overset{v}{\mathsf{T}}A(t/x), \Gamma_1}{\overset{w}{\mathsf{T}}(\forall x)A, \Gamma_2}\ {\scriptstyle v;w}$ | $\mathsf{F}\forall$ | $\dfrac{\overset{y}{\mathsf{F}}A, \Gamma_1}{\overset{z}{\mathsf{F}}(\forall x)A, \Gamma_2}\ {\scriptstyle y;z}$ |
| $\mathsf{T}\exists$ | $\dfrac{\overset{v}{\mathsf{T}}A, \Gamma_1}{\overset{w}{\mathsf{T}}(\exists x)A, \Gamma_2}\ {\scriptstyle v;w}$ | $\mathsf{F}\exists$ | $\dfrac{\overset{y}{\mathsf{F}}A(t/x), \Gamma_1}{\overset{z}{\mathsf{F}}(\exists x)A, \Gamma_2}\ {\scriptstyle y;z}$ |

Table 1: The inference figures of the classical tableau calculus **C1**.

e.g. in a right ∀-inference

$$\frac{\overset{y}{\mathsf{F}}A, \Gamma_1}{\mathsf{F}\overset{z}{(\forall x)}A, \Gamma_2}\ \mathsf{F}\forall\ y;z \quad , \tag{2}$$

the variable $x$ must not occur free in any formula in the context $\Gamma_1$.

The above assumption $z{:}\mathsf{F}(\forall x)A$ may however occur in the context $\Gamma_2$. This is a direct consequence of the third paragraph of Section 2 and applies to inferences in general.

### 3.2 On the lack of eigenvariables

Eigenvariables and their like are not only superfluous, but the lack thereof is in agreement with type theory. Their rôle as to admit a choice of what variable to bind in a Gentzen-styled inference like

$$\frac{\mathsf{F}\overset{y}{A}(a)}{\mathsf{F}\overset{z}{(\forall x)}A(x)}\ \mathsf{F}\forall\ y;z \tag{3}$$

is here absorbed by identity of tableau sequents.

To see how this comes about, suppose that the variable $u$ does not occur free in the formula $A$ of inference (2). Then, since an admissible change of bound variables does not alter the identity of a tableau sequent, the tableau sequent $z{:}\mathsf{F}(\forall u)A(u/x), \Gamma_2$ equals the conclusion of inference (2), which consequently may as well be written as

$$\frac{\overset{y}{\mathsf{F}}A, \Gamma_1}{\mathsf{F}\overset{z}{(\forall u)}A(u/x), \Gamma_2}\ \mathsf{F}\forall\ y;z \quad . \tag{4}$$

### 3.3 Weakening, contraction, and interchange

The restrictions on contexts and variables (Sect. 3.1) still permit weakening to occur at the end of each inference, within its conclusion, much like in natural deduction. This makes the restriction, that we only permit one premiss context per logical inference, negligible, because each premiss is also the conclusion of some inference, where weakening may occur.

In contrast to weakening, which depends upon our choice of inference figures, contraction and interchange only depend upon the conditions under which two tableau sequents are equal: To know that two tableau sequents,

$\Gamma_1$ and $\Gamma_2$, are equal is, per definition, to know that an object of $\Gamma_1$ is also an object of $\Gamma_2$ and, moreover, equal objects of $\Gamma_1$ are also equal objects of $\Gamma_2$, and vice versa.

We can accomplish contraction in two different ways by inferences like

$$\dfrac{\dfrac{\begin{matrix} U \\ \overset{u}{\mathsf{T}A}, \overset{w'}{\mathsf{T}A\&B} \end{matrix}}{\overset{w}{\mathsf{T}A\&B}, \overset{w'}{\mathsf{T}A\&B}}\ {\scriptstyle \mathsf{T}\&\ 1;u,w} \qquad \dfrac{}{\overset{w'}{\mathsf{T}A\&B}, \overset{z}{\mathsf{F}A\&B}}\ {\scriptstyle axiom\ w',z}}{\overset{w'}{\mathsf{T}A\&B}}\ {\scriptstyle cut\ w,z} \tag{5}$$

and

$$\dfrac{\begin{matrix} U \\ \overset{u}{\mathsf{T}A}, \overset{w'}{\mathsf{T}A\&B} \end{matrix}}{\overset{w'}{\mathsf{T}A\&B}}\ {\scriptstyle \mathsf{T}\&\ 1;u,w'} \quad , \tag{6}$$

but only the latter depends upon identity of tableau sequents. To be precise, the agreement of the third paragraph of Section 2 identifies tableau sequents like $w'$:$\mathsf{T}A\&B$ and $w'$:$\mathsf{T}A\&B, w'$:$\mathsf{T}A\&B$.

Interchange on the other hand, depends explicitly on the interpretation of tableau sequents as categories in the sense of type theory. Note that interchange permutes the assumptions of a tableau sequent, but does not alter its objects, nor the conditions under which two of its objects are equal. It therefore does not alter the tableau sequent, but only its representation, which makes interchange unnecessary.

### 3.4 Identity of derivations

Given that each inference designates an object of its conclusion category in terms of the objects of its premiss categories, we have that each derivation inductively designates an object as well. We shall consider two derivations within **C1** of the same tableau sequent to be equal if and only if they designate syntactic equal objects up to an admissible change of bound variables. Consequently, we have to complete each inference figure of **C1** with the additional information of how each inference of that inference figure designates an object of its conclusion category in terms of the objects of its premiss categories.

We shall in what follows shorten our notation and not write out contexts of variables over the set of individuals.

Completed with the additional information the inference figures of axiom and cut becomes

$$\frac{}{x(u){:}\mathsf{T}\Psi\ (u{:}\mathsf{T}A, x{:}\mathsf{F}A, \Gamma)}\ {\scriptstyle axiom\ u,x} \tag{7}$$

and

$$\frac{a{:}\mathsf{T}\Psi\ (u{:}\mathsf{T}A, \Gamma_1) \quad b{:}\mathsf{T}\Psi\ (x{:}\mathsf{F}B, \Gamma_2)}{b((u)a/x){:}\mathsf{T}\Psi\ (\Gamma_3)}\ {\scriptstyle cut\ u,x} \tag{8}$$

respectively. The substitution of the cut inference figure should be read as an explicit substitution; for related matter on explicit substitution, see [7, 9].

To complete each logical inference figure with the necessary additional information we have to introduce a corresponding constant $\mathsf{T}\&\ 1$, $\mathsf{T}\&\ 2$, $\mathsf{F}\&$, etc. Each such constant denotes an indeterminate function that designates an object of the conclusion category in terms of abstraction and the objects of the premiss categories. Completed with the necessary additional information the right &-inference figure becomes

$$\frac{a{:}\mathsf{T}\Psi\ (x{:}\mathsf{F}A, \Gamma_1) \quad b{:}\mathsf{T}\Psi\ (y{:}\mathsf{F}B, \Gamma_1)}{\mathsf{F}\&((x)a, (y)b, z){:}\mathsf{T}\Psi\ (z{:}\mathsf{F}A\&B, \Gamma_2)}\ {\scriptstyle \mathsf{F}\&\ x,y;z}\ . \tag{9}$$

The other logical inference figure cases are similar.

We have that each derivation within the classical tableau calculus inductively designates an object built up from constants and variables through abstraction, application, and explicit substitution. It thus makes sense to speak about the identity of derivations within **C1** of a tableau sequent $\Gamma$ and, moreover, the category $\tilde{\Gamma}$ of derivations within **C1** of a tableau sequent $\Gamma$. This will be important later in Section 7.1.

### 3.5 Law of the excluded middle

We can derive a tableau sequent of the form $z{:}\mathsf{F}A\vee{\sim}A, \Gamma$ by

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\overset{u}{\mathsf{T}A}, \overset{y}{\mathsf{F}A}, \Gamma}\ {\scriptstyle axiom\ u,y}}{\overset{u}{\mathsf{T}A}, \overset{z}{\mathsf{F}A\vee{\sim}A}, \Gamma}\ {\scriptstyle \mathsf{F}\vee\ 1;y,z}}{\overset{x}{\mathsf{F}{\sim}A}, \overset{z}{\mathsf{F}A\vee{\sim}A}, \Gamma}\ {\scriptstyle \mathsf{F}{\sim}\ u;x}}{\overset{z}{\mathsf{F}A\vee{\sim}A}, \Gamma}\ {\scriptstyle \mathsf{F}\vee\ 2;x,z}\ . \tag{10}$$

According to our interpretation of Section 2.2 of the classical consequence relation, to judge $z{:}\mathsf{F}A\vee{\sim}A, \Gamma$ inhabited is noting but to classically judge

$A\vee\sim A$ true under the assumptions of $\Gamma$. Consequently, derivation 10 is a derivation of the law of the excluded middle.

Note that there are only three cut free derivations of the tableau sequent $z{:}\mathsf{F}A\vee\sim A, \Gamma$ up to identity of derivations.

# 4 Decorations and decorated derivations

We shall shortly give a normalization algorithm applicable to **C1**, and then later, prove it strongly normalizing. This however requires that the normalization algorithm applies to certain decorated derivations. Consequently, those derivations must be defined before we can proceed with the algorithm itself. We make the two simultaneous definitions:

1. A *decoration* is a tuple of decorations and decorated derivations alike. We shall consider two decorations to be equal if and only if they have the same number of components and, moreover, their components are equal in pairs.

2. A *decorated derivation* is a derivation within **C1** where each inference carry a decoration by juxtaposition. We shall consider two decorated derivations of the same tableau sequent to be equal if and only if their underlying derivations are equal and, moreover, their decorations are equal in pairs.

It thus makes sense to speak about the category $\tilde{\Gamma}$ of decorated derivations within **C1** of a tableau sequent $\Gamma$. This will again be important later in Section 7.1.

Note that we use the same notation to denote both categories of derivations and categories of decorated derivations. We shall also use the noun *derivation* to denote both derivations and decorated derivations. What we mean will always be clear from the context.

We will in general denote indeterminate decorations by lower-case Greek letters.

# 5 Normalization

We shall describe a normalization algorithm able to replace each derivation by a cut free derivation that ends by the same sequent. The algorithm works for undecorated derivations as well: just ignore the decorations.

The algorithm consists of 347 different moves divided into nine families, but fortunately it is enough to consider one demonstrative move from each family to grasp the principle of the other moves as well. For the sake of clarity we will consider 14 moves: one move from each of the first eight families and six moves from the last family.

Each move allow us to replace each derivation that fits a pattern by a derivation determined by the move and the pattern; the derivation that we replace can be part of another derivation and or some decoration. Each pattern is made up of a cut and some conditions on the last inference of its right and left premiss derivation respectively, from which the large number of moves originate. These patterns are disjointed. Whenever we can choose between different moves to perform, then the choice is free.

## 5.1 Terminology

The patterns are efficiently described using the following terminology for the right premiss derivation cases. The left premiss derivation cases are similar.

1. If a cuts right premiss derivation does not end by another cut, then we say that the cut is right *inviting*.

2. If a right inviting cuts right marked cut formula equals the marked main formula of the cuts right premiss derivations last inference, then we say that the right premiss derivation is *active*, else we say that the right premiss derivation is *passive*.

3. If a right inviting cuts right marked cut formula occurs in no premiss of the cuts right premiss derivations last inference, then we say that the right premiss derivation is *critical*, else we say that the right premiss derivation is *noncritical*.

Note that if a right inviting cuts right premiss derivation is an axiom, then the right premiss derivation automatically becomes critical, and conversely, if a right inviting cuts right premiss derivation is noncritical, then the cuts right premiss derivation must end by a logical inference.

We shall shorten this terminology and say that a premiss has a property even when it is the corresponding premiss derivation that has the property.

## 5.2 Case analysis

We shall use the following case analysis whenever we need to decide what can be done with a given cut.

If a cut is not right inviting, then there is no move possible. If a cut is right inviting, then we divide the analysis into four cases depending on its right premiss derivation:

- **right premiss passive and noncritical** – Since the right premiss is noncritical, the right premiss derivation must end by a logical inference, and we can shift the cut relative this logical inference, see Section 5.3.

- **right premiss passive and critical** – We can eliminate the cut, because the right cut formula belongs to the conclusion of, but to no premiss of, the right premiss derivations last inference, see Section 5.4.

- **right premiss active and noncritical** – Since the right premiss is noncritical, the right premiss derivation must end by a logical inference, and we can distribute the cut over this logical inference, see Section 5.5.

- **right premiss active and critical** – We need to perform further case analysis, and therefor we summarize the conditions so far by defining a $\mathsf{F}$-*cut* to be a right inviting cut whose right premiss is both active and critical.

If a $\mathsf{F}$-cut is not left inviting, then there is no move possible. If a $\mathsf{F}$-cut is left inviting, then we divide the analysis into four cases depending on its left premiss derivation:

- **left premiss passive and noncritical** – Since the left premiss is noncritical, the left premiss derivation must end by a logical inference, and we can shift the cut relative this logical inference, see Section 5.6.

- **left premiss passive and critical** – We can eliminate the cut, because the left cut formula belongs to the conclusion of, but to no premiss of, the left premiss derivations last inference, see Section 5.7.

- **left premiss active and noncritical** – Since the left premiss is noncritical, the left premiss derivation must end by a logical inference, and we can distribute the cut over this logical inference, see Section 5.8.

- **left premiss active and critical** – We need to perform further case analysis, and therefor we summarize the conditions so far by defining a $\mathsf{T}$-*cut* to be a left inviting $\mathsf{F}$-cut whose left premiss is both active and critical.

We divide the analysis of a $\mathsf{T}$-cut into three cases depending on its left and right premiss derivation respectively. The cases are given mnemonic but not fully descriptive names, as opposed to the fully descriptive names used in the previous two parts of the analysis.

- **left premiss derivation an axiom** – If the left premiss derivation is an axiom, then we can eliminate the cut, see Section 5.9.
- **right premiss derivation an axiom** – If the left premiss derivation ends by a logical inference and the right premiss derivation is an axiom, then we can eliminate the cut, see Section 5.10.
- **logical cut** – If the left premiss derivation ends by a logical inference and the right premiss derivation ends by a logical inference, then we can eliminate these two inferences, see Section 5.11.

## 5.3 Right inviting cut, right premiss passive and noncritical

There are 14 moves possible for a right inviting cut where the right premiss is both passive and noncritical: one move for each logical inference figure. It is enough to consider Move 1 to understand the other moves as well.

**Move 1.** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\overset{W}{\overset{w}{\mathsf{T}}A,\Gamma_1} \qquad \dfrac{\overset{X'}{\overset{z}{\mathsf{F}}A,\overset{x'}{\mathsf{F}}C,\Gamma_2} \quad \overset{Y'}{\overset{z}{\mathsf{F}}A,\overset{y'}{\mathsf{F}}D,\Gamma_2}}{\overset{z}{\mathsf{F}}A,\overset{z'}{\mathsf{F}}C\&D,\Gamma_3}\ {\scriptstyle \mathsf{F}\&\ x',y';z'\ \beta}}{\overset{z'}{\mathsf{F}}C\&D,\Gamma_4}\ {\scriptstyle cut\ w,z\ \gamma} \qquad , \tag{11}$$

*where* $\overset{z}{\mathsf{F}}A$ *does not occur in* $\Gamma_2$*, by the corresponding derivation*

$$\dfrac{\overset{W+X'}{\overset{x'}{\mathsf{F}}C,\Gamma_1,\Gamma_2} \quad \overset{W+Y'}{\overset{y'}{\mathsf{F}}D,\Gamma_1,\Gamma_2}}{\overset{z'}{\mathsf{F}}C\&D,\Gamma_4}\ {\scriptstyle \mathsf{F}\&\ x',y';z'\ (W,\beta,\gamma)} \qquad , \tag{12}$$

*where* $W+X'$ *and* $W+Y'$ *denote the corresponding derivations*

$$\dfrac{\overset{W}{\overset{w}{\mathsf{T}}A,\Gamma_1} \quad \overset{X'}{\overset{z}{\mathsf{F}}A,\overset{x'}{\mathsf{F}}C,\Gamma_2}}{\overset{x'}{\mathsf{F}}C,\Gamma_1,\Gamma_2}\ {\scriptstyle cut\ w,z\ ()} \tag{13}$$

*and*

$$\frac{\overset{W}{\overset{w}{\mathsf{T}}A,\Gamma_1} \qquad \overset{z}{\mathsf{F}}A,\overset{Y'}{\overset{y'}{\mathsf{F}}D},\Gamma_2}{\overset{y'}{\mathsf{F}}D,\Gamma_1,\Gamma_2}\ {\scriptstyle cut\ w,z\ ()} \qquad , \tag{14}$$

*given that neither* $\overset{x'}{\mathsf{F}}C$ *nor* $\overset{y'}{\mathsf{F}}D$ *occurs in* $\Gamma_1$*, which we can guarantee through an admissible change of bound variables.*

### 5.4 Right inviting cut, right premiss passive and critical

There are 15 moves possible for a right inviting cut where the right premiss is both passive and critical: one move for the axiom inference figure, and one move for each logical inference figure. It is enough to consider Move 2 to understand the other moves as well.

**Move 2.** *We are allowed to replace each derivation that fits the pattern*

$$\frac{\overset{W}{\overset{w}{\mathsf{T}}A,\Gamma_1} \qquad \overset{z}{\mathsf{F}}A,\ \dfrac{\overset{X'}{\overset{x'}{\mathsf{F}}C,\Gamma_2} \qquad \overset{Y'}{\overset{y'}{\mathsf{F}}D,\Gamma_2}}{\overset{z'}{\mathsf{F}}C\&D,\Gamma_3}\ {\scriptstyle \mathsf{F}\&\ x',y';z'\ \beta}}{\overset{z'}{\mathsf{F}}C\&D,\Gamma_4}\ {\scriptstyle cut\ w,z\ \gamma} \qquad , \tag{15}$$

*where* $\overset{z}{\mathsf{F}}A$ *does not occur in* $\Gamma_2$*, by the corresponding derivation*

$$\frac{\overset{X'}{\overset{x'}{\mathsf{F}}C,\Gamma_2} \qquad \overset{Y'}{\overset{y'}{\mathsf{F}}D,\Gamma_2}}{\overset{z'}{\mathsf{F}}C\&D,\Gamma_4}\ {\scriptstyle \mathsf{F}\&\ x',y';z'\ (W,\beta,\gamma)} \qquad . \tag{16}$$

Among the above-mentioned 15 moves, consider the case when the right premiss derivation is an axiom, by which we are allowed to replace each derivation that fits the pattern

$$\frac{\overset{W}{\overset{w}{\mathsf{T}}A,\Gamma_1} \qquad \dfrac{}{\overset{z}{\mathsf{F}}A,\overset{w'}{\mathsf{T}}C,\overset{z'}{\mathsf{F}}C,\Gamma_3}\ {\scriptstyle axiom\ w',z'\ \beta}}{\overset{w'}{\mathsf{T}}C,\overset{z'}{\mathsf{F}}C,\Gamma_4}\ {\scriptstyle cut\ w,z\ \gamma} \tag{17}$$

by the corresponding derivation

$$\dfrac{}{\overset{w'}{\mathsf{T}}C, \overset{z'}{\mathsf{F}}C, \Gamma_4}\ \scriptstyle{axiom\ w',z'\ (W,\beta,\gamma)} \qquad . \tag{18}$$

We can validate this move by that the underlying derivations of derivation (17) and derivation (18) designate definitional identical objects in terms of the axiom and cut inference figures of Section 3.4.

## 5.5 Right inviting cut, right premiss active and noncritical

There are seven moves possible for a right inviting cut where the right premiss is both active and noncritical: one move for each right logical inference figure. It is enough to consider Move 3 to understand the other moves as well. Note that the cut of derivation (20) is a $\mathsf{F}$-cut.

**Move 3.** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\begin{matrix} W \\ \overset{w}{\mathsf{T}}A\&B, \Gamma_1 \end{matrix} \qquad \dfrac{\begin{matrix} X \\ \overset{x}{\mathsf{F}}A, \overset{z}{\mathsf{F}}A\&B, \Gamma_2 \end{matrix} \quad \begin{matrix} Y \\ \overset{y}{\mathsf{F}}A, \overset{z}{\mathsf{F}}A\&B, \Gamma_2 \end{matrix}}{\overset{z}{\mathsf{F}}A\&B, \Gamma_2}\ \scriptstyle{\mathsf{F}\&\ x,y;z\ \beta}}{\Gamma_3}\ \scriptstyle{cut\ w,z\ \gamma} \tag{19}$$

*by the corresponding derivation*

$$\dfrac{\begin{matrix} W \\ \overset{w}{\mathsf{T}}A\&B, \Gamma_1 \end{matrix} \qquad \dfrac{\begin{matrix} W+X \\ \overset{x}{\mathsf{F}}A, \Gamma_3 \end{matrix} \quad \begin{matrix} W+Y \\ \overset{y}{\mathsf{F}}A, \Gamma_3 \end{matrix}}{\overset{z}{\mathsf{F}}A\&B, \Gamma_3}\ \scriptstyle{\mathsf{F}\&\ x,y;z\ \beta}}{\Gamma_3}\ \scriptstyle{cut\ w,z\ \gamma} \qquad , \tag{20}$$

*where $W+X$ and $W+Y$ denote the corresponding derivations*

$$\dfrac{\begin{matrix} W \\ \overset{w}{\mathsf{T}}A\&B, \Gamma_1 \end{matrix} \quad \begin{matrix} X \\ \overset{x}{\mathsf{F}}A, \overset{z}{\mathsf{F}}A\&B, \Gamma_2 \end{matrix}}{\overset{x}{\mathsf{F}}A, \Gamma_3}\ \scriptstyle{cut\ w,z\ ()} \tag{21}$$

*and*

$$\dfrac{\begin{matrix} W \\ \overset{w}{\mathsf{T}}A\&B, \Gamma_1 \end{matrix} \quad \begin{matrix} Y \\ \overset{y}{\mathsf{F}}B, \overset{z}{\mathsf{F}}A\&B, \Gamma_2 \end{matrix}}{\overset{x}{\mathsf{F}}A, \Gamma_3}\ \scriptstyle{cut\ w,z\ ()} \qquad , \tag{22}$$

*given that neither $\overset{x}{\mathsf{F}}A$ nor $\overset{y}{\mathsf{F}}B$ occurs in $\Gamma_1$, which we can guarantee through an admissible change of bound variables.*

### 5.6 Left inviting F-cut, left premiss passive and noncritical

There are $14 \times 8 = 112$ moves possible for a left inviting F-cut where the left premiss is both passive and noncritical: one move for each pair of a logical inference figure and either the axiom inference figure, or a right logical inference figure. It is enough to consider Move 4 to understand the other moves as well.

**Move 4.** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\begin{array}{c}U'\\ \mathsf{T}\overset{w}{A\&B},\mathsf{T}\overset{u'}{C},\Gamma_1\end{array}}{\mathsf{T}\overset{w}{A\&B},\mathsf{T}\overset{w'}{C\&D},\Gamma_2}\,\mathsf{T\&}\ 1;u';w'\ \alpha \qquad \dfrac{\begin{array}{c}X\\ \mathsf{F}\overset{x}{A},\Gamma_3\end{array}\quad \begin{array}{c}Y\\ \mathsf{F}\overset{y}{B},\Gamma_3\end{array}}{\mathsf{F}\overset{z}{A\&B},\Gamma_4}\,\mathsf{F\&}\ x,y;z\ \beta}{\mathsf{T}\overset{w'}{C\&D},\Gamma_5}\,cut\ w,z\ \gamma \quad , \tag{23}$$

*where* $\mathsf{T}\overset{w}{A\&B}$ *does not occur in* $\Gamma_1$*, by the corresponding derivation*

$$\dfrac{\begin{array}{c}U'+Z\\ \mathsf{T}\overset{u'}{C},\Gamma_1,\Gamma_4\end{array}}{\mathsf{T}\overset{w'}{C\&D},\Gamma_5}\,\mathsf{T\&}\ 1;u';w'\ (\alpha,\beta,\gamma) \quad , \tag{24}$$

*where* $U'+Z$ *denotes the corresponding derivation*

$$\dfrac{\begin{array}{c}U'\\ \mathsf{T}\overset{w}{A\&B},\mathsf{T}\overset{u'}{C},\Gamma_1\end{array} \qquad \dfrac{\begin{array}{c}X\\ \mathsf{F}\overset{x}{A},\Gamma_3\end{array}\quad \begin{array}{c}Y\\ \mathsf{F}\overset{y}{B},\Gamma_3\end{array}}{\mathsf{F}\overset{z}{A\&B},\Gamma_4}\,\mathsf{F\&}\ x,y;z\ \beta}{\mathsf{T}\overset{u'}{C},\Gamma_1,\Gamma_4}\,cut\ w,z\ () \quad , \tag{25}$$

*given that* $\mathsf{T}\overset{u'}{C}$ *does not occur in* $\Gamma_4$*, which we can guarantee through an admissible change of bound variables.*

### 5.7 Left inviting F-cut, left premiss passive and critical

There are $15 \times 8 = 120$ moves possible for a left inviting F-cut where the left premiss is both passive and critical: one move for each pair of an inference figure, except cut, and either the axiom inference figure, or a right logical inference figure. It is enough to consider Move 5 to understand the other moves as well.

**Move 5.** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\begin{matrix}U'\\ \overset{u'}{\mathsf{T}C},\Gamma_1\end{matrix}}{\overset{w}{\mathsf{T}A\&B},\overset{w'}{\mathsf{T}C\&D},\Gamma_2}\ \mathsf{T}\&\ 1;u';w'\ \alpha \qquad \dfrac{\begin{matrix}X\\ \overset{x}{\mathsf{F}A},\Gamma_3\end{matrix}\quad\begin{matrix}Y\\ \overset{y}{\mathsf{F}B},\Gamma_3\end{matrix}}{\overset{z}{\mathsf{F}A\&B},\Gamma_4}\ \mathsf{F}\&\ x,y;z\ \beta}{\overset{w'}{\mathsf{T}C\&D},\Gamma_5}\ cut\ w,z\ \gamma \quad , \tag{26}$$

*where* $\overset{w}{\mathsf{T}A\&B}$ *does not occur in* $\Gamma_1$*, by the corresponding derivation*

$$\dfrac{\begin{matrix}U'\\ \overset{u'}{\mathsf{T}C},\Gamma_1\end{matrix}}{\overset{w'}{\mathsf{T}C\&D},\Gamma_5}\ \mathsf{T}\&\ 1;u';w'\ (\alpha,X,Y,\beta,\gamma) \quad . \tag{27}$$

## 5.8 Left inviting F-cut, left premiss active and noncritical

There are $7 \times 8 = 56$ moves possible for a left inviting F-cut where the left premiss is both active and noncritical: one move for each pair of a left logical inference figure and either the axiom inference figure, or a right logical inference figure. It is enough to consider Move 6 to understand the other moves as well. Note that the cut of derivation (29) is a T-cut.

**Move 6.** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\begin{matrix}U\\ \overset{u}{\mathsf{T}A},\overset{w}{\mathsf{T}A\&B},\Gamma_1\end{matrix}}{\overset{w}{\mathsf{T}A\&B},\Gamma_1}\ \mathsf{T}\&\ 1;u;w\ \alpha \qquad \dfrac{\begin{matrix}X\\ \overset{x}{\mathsf{F}A},\Gamma_2\end{matrix}\quad\begin{matrix}Y\\ \overset{y}{\mathsf{F}B},\Gamma_2\end{matrix}}{\overset{z}{\mathsf{F}A\&B},\Gamma_2}\ \mathsf{F}\&\ x,y;z\ \beta}{\Gamma_3}\ cut\ w,z\ \gamma \tag{28}$$

*by the corresponding derivation*

$$\dfrac{\dfrac{\begin{matrix}U+Z\\ \overset{u}{\mathsf{T}A},\Gamma_3\end{matrix}}{\overset{w}{\mathsf{T}A\&B},\Gamma_3}\ \mathsf{T}\&\ 1;u;w\ \alpha \qquad \dfrac{\begin{matrix}X\\ \overset{x}{\mathsf{F}A},\Gamma_2\end{matrix}\quad\begin{matrix}Y\\ \overset{y}{\mathsf{F}B},\Gamma_2\end{matrix}}{\overset{z}{\mathsf{F}A\&B},\Gamma_2}\ \mathsf{F}\&\ x,y;z\ \beta}{\Gamma_3}\ cut\ w,z\ \gamma \tag{29}$$

*where $U + Z$ denotes the corresponding derivation*

$$\dfrac{\overset{U}{\overset{u}{\mathsf{T}}A, \overset{w}{\mathsf{T}}A\&B, \Gamma_1} \qquad \dfrac{\overset{X}{\overset{x}{\mathsf{F}}A, \Gamma_2} \quad \overset{Y}{\overset{y}{\mathsf{F}}B, \Gamma_2}}{\overset{z}{\mathsf{F}}A\&B, \Gamma_2}\ \mathsf{F}\&\ x,y;z\ \beta}{\overset{u}{\mathsf{T}}A, \Gamma_3}\ \mathit{cut}\ w,z\ () \quad , \tag{30}$$

*given that $\overset{u}{\mathsf{T}}A$ does not occur in $\Gamma_2$, which we can guarantee through an admissible change of bound variables.*

## 5.9 T-cut, left premiss derivation an axiom

There are eight moves possible for a T-cut where the left premiss derivation is an axiom: one move for the axiom inference figure, and one move for each right logical inference figure. It is enough to consider Move 7 to understand the other moves as well.

**Move 7.** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{}{\overset{w}{\mathsf{T}}A\&B, \overset{z'}{\mathsf{F}}A\&B, \Gamma_1}\ \mathit{axiom}\ w,z'\ \alpha \qquad \dfrac{\overset{X}{\overset{x}{\mathsf{F}}A, \Gamma_2} \quad \overset{Y}{\overset{y}{\mathsf{F}}B, \Gamma_2}}{\overset{z}{\mathsf{F}}A\&B, \Gamma_2}\ \mathsf{F}\&\ x,y;z\ \beta}{\overset{z'}{\mathsf{F}}A\&B, \Gamma_3}\ \mathit{cut}\ w,z\ \gamma \tag{31}$$

*by the corresponding derivation*

$$\dfrac{\overset{X}{\overset{x}{\mathsf{F}}A, \Gamma_2} \quad \overset{Y}{\overset{y}{\mathsf{F}}B, \Gamma_2}}{\overset{z'}{\mathsf{F}}A\&B, \Gamma_3}\ \mathsf{F}\&\ x,y;z'\ (\alpha,\beta,\gamma) \quad . \tag{32}$$

We can understand the corresponding move, when the right premiss derivation is an axiom, in the same manner as we understood the corresponding move of Section 5.4.

## 5.10 T-cut, right premiss derivation an axiom

There are seven moves possible for a T-cut where the left premiss derivation ends by a logical inference and the right premiss derivation is an axiom: one move for each left logical inference figure. It is enough to consider Move 8 to understand the other moves as well.

**Move 8.** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\begin{matrix}U\\ \overset{u}{\mathsf{T}}A,\Gamma_1\end{matrix}}{\overset{w}{\mathsf{T}}A\&B,\Gamma_1}\ {\scriptstyle \mathsf{T}\&\ 1;u;w\ \alpha} \qquad \dfrac{}{\overset{w'}{\mathsf{T}}A\&B,\overset{z}{\mathsf{F}}A\&B,\Gamma_2}\ {\scriptstyle axiom\ w',z\ \beta}}{\overset{w'}{\mathsf{T}}A\&B,\Gamma_3}\ {\scriptstyle cut\ w,z\ \gamma} \tag{33}$$

*by the corresponding derivation*

$$\dfrac{\begin{matrix}U\\ \overset{u}{\mathsf{T}}A,\Gamma_1\end{matrix}}{\overset{w'}{\mathsf{T}}A\&B,\Gamma_3}\ {\scriptstyle \mathsf{T}\&\ 1;u;w'\ (\alpha,\beta,\gamma)} \tag{34}$$

We could extend this family of moves, and allow the left premiss derivation to be an axiom, because the corresponding move would be in agreement with Section 5.9.

## 5.11 Logical cut

There are eight moves possible for a $\mathsf{T}$-cut where the left premiss derivation ends by a logical inference and the right premiss derivation ends by a logical inference: one move for each pair of a left logical inference figure and a corresponding right logical inference figure. It is enough to consider Move 9-14 to understand the other moves as well.

**Move 9 (&-reduction).** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\begin{matrix}U\\ \overset{u}{\mathsf{T}}A,\Gamma_1\end{matrix}}{\overset{w}{\mathsf{T}}A\&B,\Gamma_2}\ {\scriptstyle \mathsf{T}\&\ 1;u;w\ \alpha} \qquad \dfrac{\begin{matrix}X\\ \overset{x}{\mathsf{F}}A,\Gamma_3\end{matrix}\quad \begin{matrix}Y\\ \overset{y}{\mathsf{F}}B,\Gamma_3\end{matrix}}{\overset{z}{\mathsf{F}}A\&B,\Gamma_4}\ {\scriptstyle \mathsf{F}\&\ x,y;z\ \beta}}{\Gamma_5}\ {\scriptstyle cut\ w,z\ \gamma} \tag{35}$$

*by the corresponding derivation*

$$\dfrac{\begin{matrix}U\\ \overset{u}{\mathsf{T}}A,\Gamma_1\end{matrix}\quad \begin{matrix}X\\ \overset{x}{\mathsf{F}}A,\Gamma_3\end{matrix}}{\Gamma_5}\ {\scriptstyle cut\ u,x\ (\alpha,Y,\beta,\gamma)}\ . \tag{36}$$

**Move 10 ($\vee$-reduction).** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\overset{U}{\overset{u}{\mathsf{T}}A,\Gamma_1} \quad \overset{V}{\overset{v}{\mathsf{T}}B,\Gamma_1}}{\overset{w}{\mathsf{T}}A\vee B,\Gamma_2}\ \mathsf{T}\vee\ u,v;w\ \alpha \qquad \dfrac{\overset{X}{\overset{x}{\mathsf{F}}A,\Gamma_3}}{\overset{z}{\mathsf{F}}A\vee B,\Gamma_4}\ \mathsf{F}\vee\ 1;x;z\ \beta}{\Gamma_5}\ \mathit{cut}\ w,z\ \gamma \tag{37}$$

*by the corresponding derivation*

$$\dfrac{\overset{U}{\overset{u}{\mathsf{T}}A,\Gamma_1} \quad \overset{X}{\overset{x}{\mathsf{F}}A,\Gamma_3}}{\Gamma_5}\ \mathit{cut}\ u,x\ (V,\alpha,\beta,\gamma). \tag{38}$$

**Move 11 ($\supset$-reduction).** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\overset{X}{\overset{x}{\mathsf{F}}A,\Gamma_1} \quad \overset{V}{\overset{v}{\mathsf{T}}B,\Gamma_1}}{\overset{w}{\mathsf{T}}A\supset B,\Gamma_2}\ \mathsf{T}\supset\ x,v;w\ \alpha \qquad \dfrac{\overset{UY}{\overset{u}{\mathsf{T}}A,\overset{y}{\mathsf{F}}B,\Gamma_3}}{\overset{z}{\mathsf{F}}A\supset B,\Gamma_4}\ \mathsf{F}\supset\ u,y;z\ \beta}{\Gamma_5}\ \mathit{cut}\ w,z\ \gamma \tag{39}$$

*by the corresponding derivation*

$$\dfrac{\dfrac{\overset{V}{\overset{v}{\mathsf{T}}B,\Gamma_1} \quad \overset{UY}{\overset{u}{\mathsf{T}}A,\overset{y}{\mathsf{F}}B,\Gamma_3}}{\overset{u}{\mathsf{T}}A,\Gamma_1,\Gamma_3}\ \mathit{cut}\ v,y\ () \qquad \overset{X}{\overset{x}{\mathsf{F}}A,\Gamma_1}}{\Gamma_5}\ \mathit{cut}\ u,x\ (\alpha,\beta,\gamma). \tag{40}$$

**Move 12 ($\sim$-reduction).** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\overset{X}{\overset{x}{\mathsf{F}}A,\Gamma_1}}{\overset{w}{\mathsf{T}}{\sim}A,\Gamma_2}\ \mathsf{T}{\sim}\ x;w\ \alpha \qquad \dfrac{\overset{U}{\overset{u}{\mathsf{T}}A,\Gamma_3}}{\overset{z}{\mathsf{F}}{\sim}A,\Gamma_4}\ \mathsf{F}{\sim}\ u;z\ \beta}{\Gamma_5}\ \mathit{cut}\ w,z\ \gamma \tag{41}$$

*by the corresponding derivation*

$$\dfrac{\overset{U}{\overset{u}{\mathsf{T}}A,\Gamma_1} \quad \overset{X}{\overset{x}{\mathsf{F}}A,\Gamma_3}}{\Gamma_5}\ \mathit{cut}\ u,x\ (\alpha,\beta,\gamma). \tag{42}$$

**Move 13 (∀-reduction).** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\overset{V}{\overset{v}{\mathsf{T}}A(t/x),\Gamma_1}}{\overset{w}{\mathsf{T}}(\forall x)A,\Gamma_2}\;{\scriptstyle \mathsf{T}\forall\ v;w\ \alpha}\qquad \dfrac{\overset{Y}{\overset{y}{\mathsf{F}}A,\Gamma_3}}{\overset{z}{\mathsf{F}}(\forall x)A,\Gamma_4}\;{\scriptstyle \mathsf{F}\forall\ y;z\ \beta}}{\Gamma_5}\;{\scriptstyle cut\ w,z\ \gamma} \tag{43}$$

*by the corresponding derivation*

$$\dfrac{\overset{V}{\overset{v}{\mathsf{T}}A(t/x),\Gamma_1}\qquad \overset{Y(t/x)}{\overset{y}{\mathsf{F}}A(t/x),\Gamma_3}}{\Gamma_5}\;{\scriptstyle cut\ v,y\ (\alpha,\beta,\gamma)}\,. \tag{44}$$

**Move 14 (∃-reduction).** *We are allowed to replace each derivation that fits the pattern*

$$\dfrac{\dfrac{\overset{V}{\overset{v}{\mathsf{T}}A,\Gamma_1}}{\overset{w}{\mathsf{T}}(\exists x)A,\Gamma_2}\;{\scriptstyle \mathsf{T}\exists\ v;w\ \alpha}\qquad \dfrac{\overset{Y}{\overset{y}{\mathsf{F}}A(t/x),\Gamma_3}}{\overset{z}{\mathsf{F}}(\exists x)A,\Gamma_4}\;{\scriptstyle \mathsf{F}\exists\ y;z\ \beta}}{\Gamma_5}\;{\scriptstyle cut\ w,z\ \gamma} \tag{45}$$

*by the corresponding derivation*

$$\dfrac{\overset{V(t/x)}{\overset{v}{\mathsf{T}}A(t/x),\Gamma_1}\qquad \overset{Y}{\overset{y}{\mathsf{F}}A(t/x),\Gamma_3}}{\Gamma_5}\;{\scriptstyle cut\ v,y\ (\alpha,\beta,\gamma)}\,. \tag{46}$$

## 6 A comparison with Gentzen's Hauptsatz

Let us, to gain some hands-on experience, apply both the algorithm implicit in Gentzen's Hauptsatz and the algorithm of Section 5 without decorations to a derivation of the form

$$\dfrac{\dfrac{\overset{U}{\overset{u}{\mathsf{T}}A,\overset{w}{\mathsf{T}}A\&B,\Gamma}}{\overset{w}{\mathsf{T}}A\&B,\Gamma}\;{\scriptstyle \mathsf{T}\&\ 1;u;w}\qquad \dfrac{\overset{X}{\overset{x}{\mathsf{F}}A,\overset{z}{\mathsf{F}}A\&B,\Gamma}\quad \overset{Y}{\overset{y}{\mathsf{F}}B,\overset{z}{\mathsf{F}}A\&B,\Gamma}}{\overset{z}{\mathsf{F}}A\&B,\Gamma}\;{\scriptstyle \mathsf{F}\&\ x,y;z}}{\Gamma}\;{\scriptstyle cut\ w,z}\,. \tag{47}$$

## 6.1 Normalization according to Gentzen's Hauptsatz

We now reduce derivation (47) according to the algorithm implicit in Gentzen's Hauptsatz. First, replace it by the derivation

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{U+Z}{\overset{u}{\mathsf{T}}A,\Gamma}}{\overset{w}{\mathsf{T}}A\&B,\Gamma}\,{\scriptstyle \mathsf{T}\&\ 1;u;w} \qquad \dfrac{\genfrac{}{}{0pt}{}{W+X}{\overset{x}{\mathsf{F}}A,\Gamma}\quad \genfrac{}{}{0pt}{}{W+Y}{\overset{y}{\mathsf{F}}B,\Gamma}}{\overset{z}{\mathsf{F}}A\&B,\Gamma}\,{\scriptstyle \mathsf{F}\&\ x,y;z}}{\Gamma}\,{\scriptstyle cut\ w,z} \quad , \tag{48}$$

where $U+Z$, $W+X$, and $W+Y$ are the derivations

$$\dfrac{\genfrac{}{}{0pt}{}{U}{\overset{u}{\mathsf{T}}A,\overset{w}{\mathsf{T}}A\&B,\Gamma} \qquad \dfrac{\genfrac{}{}{0pt}{}{X}{\overset{x}{\mathsf{F}}A,\overset{z}{\mathsf{F}}A\&B,\Gamma}\quad \genfrac{}{}{0pt}{}{Y}{\overset{y}{\mathsf{F}}B,\overset{z}{\mathsf{F}}A\&B,\Gamma}}{\overset{z}{\mathsf{F}}A\&B,\Gamma}\,{\scriptstyle \mathsf{F}\&\ x,y;z}}{\overset{u}{\mathsf{T}}A,\Gamma}\,{\scriptstyle cut\ w,z} \quad , \tag{49}$$

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{U}{\overset{u}{\mathsf{T}}A,\overset{w}{\mathsf{T}}A\&B,\Gamma}}{\overset{w}{\mathsf{T}}A\&B,\Gamma}\,{\scriptstyle \mathsf{T}\&\ 1;u;w} \qquad \genfrac{}{}{0pt}{}{X}{\overset{x}{\mathsf{F}}A,\overset{z}{\mathsf{F}}A\&B,\Gamma}}{\overset{x}{\mathsf{F}}A,\Gamma}\,{\scriptstyle cut\ w,z} \quad , \tag{50}$$

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{U}{\overset{u}{\mathsf{T}}A,\overset{w}{\mathsf{T}}A\&B,\Gamma}}{\overset{w}{\mathsf{T}}A\&B,\Gamma}\,{\scriptstyle \mathsf{T}\&\ 1;u;w} \qquad \genfrac{}{}{0pt}{}{Y}{\overset{y}{\mathsf{F}}B,\overset{z}{\mathsf{F}}A\&B,\Gamma}}{\overset{y}{\mathsf{F}}B,\Gamma}\,{\scriptstyle cut\ w,z} \quad . \tag{51}$$

Second, apply Move 9 and replace derivation (48) by the derivation

$$\dfrac{\genfrac{}{}{0pt}{}{U+Z}{\overset{u}{\mathsf{T}}A,\Gamma} \quad \genfrac{}{}{0pt}{}{W+X}{\overset{x}{\mathsf{F}}A,\Gamma}}{\Gamma}\,{\scriptstyle cut\ u,x} \quad . \tag{52}$$

Expansion of $U+Z$ gives

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{U}{\overset{u}{\mathsf{T}}A,\overset{w}{\mathsf{T}}A\&B,\Gamma} \quad \genfrac{}{}{0pt}{}{Z}{\overset{z}{\mathsf{F}}A\&B,\Gamma}}{\overset{u}{\mathsf{T}}A,\Gamma}\,{\scriptstyle cut\ w,z} \qquad \genfrac{}{}{0pt}{}{W+X}{\overset{x}{\mathsf{F}}A,\Gamma}}{\Gamma}\,{\scriptstyle cut\ u,x} \quad , \tag{53}$$

where $Z$ is the derivation

$$\cfrac{\overset{X}{\overset{x}{\mathsf{F}}A, \overset{z}{\mathsf{F}}A\&B, \Gamma} \qquad \overset{Y}{\overset{y}{\mathsf{F}}B, \overset{z}{\mathsf{F}}A\&B, \Gamma}}{\overset{z}{\mathsf{F}}A\&B, \Gamma}\ \mathsf{F}\&\ x,y;z \quad . \tag{54}$$

## 6.2 Normalization according to Section 5

We now reduce derivation (47) according to the algorithm of Section 5. First, apply Move 3 and replace it by the derivation

$$\cfrac{\cfrac{\overset{U}{\overset{u}{\mathsf{T}}A, \overset{w}{\mathsf{T}}A\&B, \Gamma}}{\overset{w}{\mathsf{T}}A\&B, \Gamma}\ \mathsf{T}\&\ 1;u;w \qquad \cfrac{\overset{W+X}{\overset{x}{\mathsf{F}}A, \Gamma} \qquad \overset{W+Y}{\overset{y}{\mathsf{F}}B, \Gamma}}{\overset{z}{\mathsf{F}}A\&B, \Gamma}\ \mathsf{F}\&\ x,y;z}{\Gamma}\ cut\ w,z \quad , \tag{55}$$

where $W+X$ and $W+Y$ coincide with the derivations (50) and (51) respectively. Second, apply Move 6 and replace derivation (55) by the derivation

$$\cfrac{\cfrac{\overset{U+Z'}{\overset{u}{\mathsf{T}}A, \Gamma}}{\overset{w}{\mathsf{T}}A\&B, \Gamma}\ \mathsf{T}\&\ 1;u;w \qquad \cfrac{\overset{W+X}{\overset{x}{\mathsf{F}}A, \Gamma} \qquad \overset{W+Y}{\overset{y}{\mathsf{F}}B, \Gamma}}{\overset{z}{\mathsf{F}}A\&B, \Gamma}\ \mathsf{F}\&\ x,y;z}{\Gamma}\ cut\ w,z \quad , \tag{56}$$

where $U + Z'$ is the derivation

$$\cfrac{\overset{U}{\overset{u}{\mathsf{T}}A, \overset{w}{\mathsf{T}}A\&B, \Gamma} \qquad \cfrac{\overset{W+X}{\overset{x}{\mathsf{F}}A, \Gamma} \qquad \overset{W+Y}{\overset{y}{\mathsf{F}}B, \Gamma}}{\overset{z}{\mathsf{F}}A\&B, \Gamma}\ \mathsf{F}\&\ x,y;z}{\overset{u}{\mathsf{T}}A, \Gamma}\ cut\ w,z \quad . \tag{57}$$

Third, apply Move 9 and replace derivation (56) by the derivation

$$\cfrac{\overset{U+Z'}{\overset{u}{\mathsf{T}}A, \Gamma} \qquad \overset{W+X}{\overset{x}{\mathsf{F}}A, \Gamma}}{\Gamma}\ cut\ u,x \quad . \tag{58}$$

Expansion of $U + Z'$ gives

$$\frac{\dfrac{\overset{U}{\overset{u}{\mathsf{T}A}, \overset{w}{\mathsf{T}A\&B}, \Gamma} \qquad \overset{Z'}{\overset{z}{\mathsf{F}A\&B}, \Gamma}}{\overset{u}{\mathsf{T}A}, \Gamma}\ {\scriptstyle cut\ w,z} \qquad \overset{W+X}{\overset{x}{\mathsf{F}A}, \Gamma}}{\Gamma}\ {\scriptstyle cut\ u,x}\ , \tag{59}$$

where $Z'$ is the derivation

$$\frac{\overset{W+X}{\overset{x}{\mathsf{F}A}, \Gamma} \qquad \overset{W+Y}{\overset{y}{\mathsf{F}B}, \Gamma}}{\overset{z}{\mathsf{F}A\&B}, \Gamma}\ {\scriptstyle \mathsf{F}\&\ x,y;z}\ . \tag{60}$$

Derivation (59) should be compared to derivation (53).

### 6.3 Remark

The difference between derivation (54) and derivation (60) arises only because the two premiss derivations of the cut of the initial derivation (47) both end by a contraction. If we change derivation (47) to either

$$\frac{\dfrac{\overset{U}{\overset{u}{\mathsf{T}A}, \Gamma}}{\overset{w}{\mathsf{T}A\&B}, \Gamma}\ {\scriptstyle \mathsf{T}\&\ 1;u;w} \qquad \dfrac{\overset{X}{\overset{x}{\mathsf{F}A}, \overset{z}{\mathsf{F}A\&B}, \Gamma} \qquad \overset{Y}{\overset{y}{\mathsf{F}B}, \overset{z}{\mathsf{F}A\&B}, \Gamma}}{\overset{z}{\mathsf{F}A\&B}, \Gamma}\ {\scriptstyle \mathsf{F}\&\ x,y;z}}{\Gamma}\ {\scriptstyle cut\ w,z} \tag{61}$$

or

$$\frac{\dfrac{\overset{U}{\overset{u}{\mathsf{T}A}, \overset{w}{\mathsf{T}A\&B}, \Gamma}}{\overset{w}{\mathsf{T}A\&B}, \Gamma}\ {\scriptstyle \mathsf{T}\&\ 1;u;w} \qquad \dfrac{\overset{X}{\overset{x}{\mathsf{F}A}, \Gamma} \qquad \overset{Y}{\overset{y}{\mathsf{F}B}, \Gamma}}{\overset{z}{\mathsf{F}A\&B}, \Gamma}\ {\scriptstyle \mathsf{F}\&\ x,y;z}}{\Gamma}\ {\scriptstyle cut\ w,z}\ , \tag{62}$$

then the algorithm implicit in Gentzen's Hauptsatz and the algorithm of Section 5 agree, and the difference disappears. The two algorithms furthermore agree on logical cuts (Sect. 5.11).

# 7 Normalization theorems

We shall prove that the algorithm of Section 5 is both confluent (Sect. 7.1) and strongly normalizing (Sect. 7.3). The proof of the latter conforms with

Gandy's technique [2], which the author learned from [3], and it relies on the algorithm being weakly normalizing (Sect. 7.2). The latter is proved along the same line of argument as Gentzen's Hauptsatz.

These normalization theorems are all formulated and proved up to identity of decorated derivations, but they all apply to undecorated derivations as well.

## 7.1 Confluence

We shall prove that the algorithm of Section 5 is confluent. This result amounts to Theorem 1.

Let $\Gamma$ be a tableau sequent. At the end of Section 3.4 we defined the category $\tilde{\Gamma}$ of derivations within the classical tableau calculus of $\Gamma$. We can evidently consider the category $\tilde{\Gamma}$ as a type. Hereby each inference of $\Gamma_{n+1}$ from $\Gamma_1, ..., \Gamma_n$ becomes an *inference function* from $\tilde{\Gamma}_1 \times \cdots \times \tilde{\Gamma}_n$ into $\tilde{\Gamma}_{n+1}$. We define a *derivation function* to be a possibly empty composition of inference functions linear in its arguments. We can extend this terminology to decorated derivations as well and shall henceforth take this extended terminology for granted. Note that even the decorations of a give derivation function may depend upon its arguments.

We furthermore extend the terminology of Section 5 and say that a cut of a given derivation or inference is *inviting* whenever there exists a move applicable to the cut. Since the patterns of Section 5 are disjointed an inviting cut uniquely determines its move.

Let $X$ be a derivation such that $X = f(X_1, ..., X_n)$ where $X_1$, ..., $X_n$ are derivations and $f$ is a derivation function. For each $j = 1, ..., n$ suppose that $X_j$ ends by an inviting cut by which it reduces to another derivation $Y_j$. We then say that $X$ *reduces to* $f(Y_1, ..., Y_n)$ *in one* $n$*-block*, or simply *in one block*, leaving out the number of involved cuts.

We immediately have

**Lemma 1.** *Let* $X$ *be a derivation such that* $X = f(X_1, ..., X_n)$ *where* $X_1$, ..., $X_n$ *are derivations and* $f$ *is a derivation function. For each* $j = 1, ..., n$ *suppose that* $X_j$ *reduces to another derivation* $Y_j$ *in one block. Then* $X$ *reduces to* $f(Y_1, ..., Y_n)$ *in one block.*

We now continue from before the above lemma. Suppose that $X$ ends by an inviting cut $C$ and note that the pattern of its move does not involve any other cut than $C$. Since for each $j = 1, ..., n$ the derivation $X_j$ ends by a cut, the move applicable to $C$ allows us to replace $f$ by a corresponding function $\bar{f}$ which the move dictates independent of the arguments of $f$, that is, the move applicable to $C$ allows us to replace $f(X_1, ..., X_n)$ by $\bar{f}(X_1, ..., X_n)$ independent of $X_1, ..., X_n$.

We immediately have

**Lemma 2.** *With use of the above notation, we can reduce $\bar{f}(X_1, ..., X_n)$ and $f(Y_1, ..., Y_n)$ to one joint derivation $\bar{f}(Y_1, ..., Y_n)$ in one block each.*

The next lemma contains the essential part that we need to prove confluence for the classical tableau calculus. It is simplified by the following terminology: Let $X$ be a derivation. If a given cut of $X$ is either part of one of the two premiss derivations of another given cut of $X$ or part of its decoration, then we say that the first cut is *above* the second cut.

**Lemma 3.** *If we can reduce one derivation $X$ to two derivations $F$ and $G$ in one blocks each, then we can reduce $F$ and $G$ to one joint derivation in one block each.*

*Proof.* We have $X = f(X_1, ..., X_m)$ and $X = g(X_{m+1}, ..., X_n)$ and similarly $F = f(Y_1, ..., Y_m)$ and $G = g(Y_{m+1}, ..., Y_n)$, where $f$ and $g$ are derivation functions and for each $j = 1, ..., n$ the derivation $X_j$ ends by an inviting cut $C_j$ by which it reduces to the derivation $Y_j$. We can without loss of generality assume that $C_1, ..., C_{m-p}, C_{m+1}, ..., C_{n-p}$ are distinct but $C_{m-k} = C_{n-k}$ for $k = 1, ..., p$.

Let $M = \{1, ..., m - p\}$, $N = \{m + 1, ..., n - p\}$, and $L = M \cup N$. For each $j \in L$ let $L_j = \{i \in L | C_i \text{ is above } C_j\}$ and denote its elements by $l_j(1), ..., l_j(|L_j|)$. Note that if $j \in N$ then $L_j \subset M$ and vice versa. Finally let

$$K = L \setminus \bigcup_{j \in L} L_j$$

and denote its elements by $k(1), ..., k(|K|)$.

It follows from the above definitions that for each $k \in K$ there exists a derivation function $h_k$ such that $X_k = h_k(X_{l_k(1)}, ..., X_{l_k(|L_k|)})$ and, moreover, that there exists a derivation function $h$ such that

$$X = h(X_{k(1)}, ..., X_{k(|K|)}, X_{n-p}, ..., X_n).$$

We can without loss of generality assume that $K \cap M = \{k(1), ..., k(|K \cap M|)\}$ and $K \cap N = \{k(|K \cap M| + 1), ..., k(|K|)\}$. We then have

$$\begin{aligned} F &= h(Y_{k(1)}, ..., Y_{k(|K \cap M|)}, Z_{k(|K \cap M|+1)}, ..., Z_{k(|K|)}, Y_{n-p}, ..., Y_n), \\ G &= h(Z_{k(1)}, ..., Z_{k(|K \cap M|)}, Y_{k(|K \cap M|+1)}, ..., Y_{k(|K|)}, Y_{n-p}, ..., Y_n), \end{aligned}$$

where $Z_k = h_k(Y_{l_k(1)}, ..., Y_{l_k(|L_k|)})$. The result then follows by Lemma 1 and Lemma 2. □

Let $X$ be a derivation that reduces to another derivation $Y$ in one $m$-block. If $m \geq 1$ then we say that $X$ reduces to $Y$ in one *strict* block. We shall not have to use the concept of strictness until Section 7.3, but then in the form of a *strict reduction sequence.*

We define a (strict) *reduction sequence* from a derivation $X_1$ to a derivation $X_{n+1}$ to be a sequence of $n+1$ derivations $X_1$, ..., $X_{n+1}$ such that $X_j$ reduces to $X_{j+1}$ in one (strict) block for each $j = 1, ..., n$.

In terms of the above reduction sequence, we furthermore say that $X_1$ reduces to $X_{n+1}$ in $n$ blocks. We can then formulate and prove

**Theorem 1 (Church, Rosser).** *The algorithm of Section 5 is confluent, that is, if we can reduce one derivation to two derivations $F$ and $G$ in $m$ and $n$ blocks respectively, then we can reduce $F$ and $G$ to one joint derivation in $n$ and $m$ blocks respectively.*

*Proof.* We can prove Theorem 1 by mathematical induction over $m$ and $n$ with Lemma 3 part of the induction. □

## 7.2 Weak normalization

We shall prove that the algorithm of Section 5 is weakly normalizing. This result amounts to Theorem 2.

The proof of Theorem 2 uses Gentzen's concept of the degree of a cut. The *degree* of a formula is defined to be the number of logical connectives and quantifiers occurring in it. By the *cut formula* of a cut

$$\frac{\begin{matrix} U \\ \overset{u}{\mathsf{T}}A, \Gamma_1 \end{matrix} \quad \begin{matrix} X \\ \overset{x}{\mathsf{F}}A, \Gamma_2 \end{matrix}}{\Gamma_3} \; {\scriptstyle cut\ u,x\ \alpha} \tag{63}$$

we mean the formula $A$. The *degree* of a cut $C$ is defined to be the degree of its cut formula, which we shall denote by $\mathrm{d}(C)$. We inspect the algorithm of Section 5 and conclude that

**Claim 1.** *Let $X$ be a derivation that ends by a $\mathsf{T}$-cut cut $C$. Then $C$ is inviting and each cut $C'$ that originates from the move applicable to $C$ fulfill* $\mathrm{d}(C') < \mathrm{d}(C)$.

The proof of Theorem 2 follows the same line of argument as Gentzen's proof of his Hauptsatz, but uses a refined measure of height only applicable to cuts. It is defined in terms of the usual measure of height.

**Definition 1.** *The* height *of a derivation $X$ whose last inference has the premiss derivations $X_1$, ..., $X_n$ is defined by*

$$h(X) = 1 + \max[h(X_1), ..., h(X_n)].$$

**Definition 2.** *The* relevant height *of a cut $C$ that has left premiss derivation $X_1$ and right premiss derivation $X_2$ is defined by*

$$\begin{cases} \mathrm{rh}(C) &= \mathrm{h}(X_1) + \mathrm{h}(X_2) & \text{if } C \text{ is not a } \mathsf{F}\text{-cut,} \\ \mathrm{rh}(C) &= \mathrm{h}(X_1) & \text{if } C \text{ is a } \mathsf{F}\text{-cut but not a } \mathsf{T}\text{-cut,} \\ \mathrm{rh}(C) &= 0 & \text{if } C \text{ is a } \mathsf{T}\text{-cut.} \end{cases}$$

We inspect the algorithm of Section 5 and conclude that

**Claim 2.** *Let $X$ be a derivation that ends by an inviting cut $C$. If $C$ is not a $\mathsf{T}$-cut, then each cut $C'$ that originates from the move applicable to $C$ fulfill* $\mathrm{d}(C') = \mathrm{d}(C)$ *and* $\mathrm{rh}(C') < \mathrm{rh}(C)$.

**Lemma 4.** *Let $X$ be a derivation that ends by a cut with cut free premiss derivations. Then we can reduce $X$ to a cut free derivation, which we shall denote by* $\mathrm{nf}(X)$.

*Proof.* Let $C$ denote the above-mentioned cut. We can prove Lemma 4 by mathematical induction over $\mathrm{d}(C)$ and $\mathrm{rh}(C)$ with Claim 1 and Claim 2 part of the induction. Just note that since the cut is both left and right inviting it is inviting as well. □

**Theorem 2 (Gentzen).** *The algorithm of Section 5 is weakly normalizing, that is, we can reduce each derivation $X$ to a cut free derivation, which we shall denote by* $\mathrm{nf}(X)$.

*Proof.* We can prove Theorem 1 by mathematical induction over the number of cuts of $X$ with Lemma 4 part of the induction. □

### 7.3 Strong normalization

We shall prove that the algorithm of Section 5 is strongly normalizing. This result amounts to Theorem 3.

The proof of Theorem 3 makes use of a functions that measures the size of a derivation, which corresponds to the measure of [3].

**Definition 3.** *The* size *of a derivation $X$ whose last inference has the premiss derivations $X_1$, ..., $X_n$ and carry the decoration $\alpha$ is defined by*

$$|X| = |X_1| + ... + |X_n| + |\alpha|,$$

*where the* size *of the decoration* $\alpha = (\alpha_1, ..., \alpha_m)$ *is defined by*

$$|\alpha| = |\alpha_1| + ... + |\alpha_m| + 1.$$

We inspect the algorithm of Section 5 and conclude that

**Claim 3.** *Let $X$ be a derivation that reduces to another derivation $Y$ in one strict block. Then $|X| < |Y|$.*

**Theorem 3 (Gandy).** *The algorithm of Section 5 is strongly normalizing, or more precisely, for each strict reduction sequence $X_1, ..., X_{n+1}$ its length $n$ is bounded by a function of $X_1$.*

*Proof.* We can compute $\mathrm{nf}(X_1)$ and $\mathrm{nf}(X_{n+1})$ by Theorem 2. We have

$$|X_1| < |X_2| < ... < |X_{n+1}| \leq |\mathrm{nf}(X_{n+1})|$$

by Claim 3, whence $n \leq |\mathrm{nf}(X_{n+1})|$. We furthermore have $\mathrm{nf}(X_{n+1}) = \mathrm{nf}(X_1)$ by Theorem 1. Hence $n \leq |\mathrm{nf}(X_1)|$. □

# 8 Conclusion

We have refined Gentzen's classical sequent calculus into a classical tableau calculus within the framework of Martin-Löf's type theory. This calculus has essentially the same inference figures as Gentzen's classical sequent calculus, but no structural inference figures except cut. Furthermore, it admits a normalization algorithm similar to the algorithm implicit in Gentzen's Hauptsatz, however confluent and strongly normalizing. The proof of the latter conforms with Gandy's technique, which in turn relies on the algorithm being weakly normalizing, which was proved along the same line of argument as Gentzen's Hauptsatz.

The relative simplicity of the algorithm springs from a careful choice of inference figures with respect to their contexts and, moreover, the identity of tableau sequents, which in turn intimately depends on the interpretation of sequents as categories in the sense of type theory. Together they make structural inference figures and moves for their propagation redundant.

The variables that mark the assumptions of sequents are analogous to the marks used to discharge assumptions in natural deduction, granted that they are placed upon an assumption even before that assumption is discharged. Thus they can be thought of as a book keeping device separate from type theory. On the other hand, the identity of tableau sequents and likewise derivations have, according to the author, their natural explanations within type theory, and should be considered within that framework.

The use of decorations makes it possible to adapt Gandy's technique into a form applicable to sequent calculi. It however remains an open question whether this technique can be extended to proofs incorporating a rule of induction.

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