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Filosofie licentiatavhandling

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Chapter 0

Introduction

In recent years the topic of Hardy inequalities and their applications seems to have become more and more popular. Although the original Hardy inequality, proved by G.H.Hardy, was discovered in the 1920's, new versions are stated and proved and old ones are improved almost a century later. One reason for their popularity is their usefulness in various types of applications.

It is hard to give a precise definition of when an inequality is of Hardy-type. It is a designation for a large class of inequalities including some which are known under different names such as the Friedrichs inequality.
The standard form of Hardys inequality, when we are interesting only in an estimate of the first order derivatives, is

$$\left(\int_{\Omega} |\nabla u(x)|^p v(x) dx\right)^{\frac{1}{p}} \geq \left(\int_{\Omega} |u(x)|^q w(x) dx\right)^{\frac{1}{q}} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$, $v(x)$ and $w(x)$ are weight functions and $0 < q \leq \infty$ and $1 \leq p \leq \infty$.
Of course, not all such inequalities are true. Their validness depends heavily on the relation between the parameters, the weight functions and on the function class to which u belongs.
The inequality (0.1) is usually not exact, that is, there is no function for which we have equality. Therefore it is natural to expect that some extra term might be added on the RH-side to improve the inequality. This can be done in many possible ways.
One type of inequality which is common in the litteraure is the case when the weight functions are of the form $w(dist(x, M))$ where M is some manifold or set. The particular case where M is a single point, for example the origin, has been extensively studied (see for example [19],[20]) and is by now quite well understood.
Sometimes, though, it is more natural to choose $M = \partial\Omega$ and consider functions which are zero on Ω. This case is usually more difficult to analyze, because various kinds of "reduction of dimension" techniques used when $M = \{0\}$ does not apply here. In the first chapter we will apply another such technique adapted for this situation. The main theorem in that chapter is the

Hardy inequality

$$\int_\Omega |\nabla u|^p dx \geq c_p \int_\Omega \frac{|u(x)|^p}{dist(x,\partial\Omega)^p} dx + \frac{a(p,n)}{|\Omega|^{\frac{p}{n}}} \int_\Omega |u(x)|^p dx, \quad u \in \overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega), \tag{0.2}$$

for convex domains Ω, where c_p is the optimal constant and $a(p,n) > 0$.

The second chapter contains various modifications and generalizations of the results in chapter 1. For example we generalize (0.2) by allowing vector valued functions(v.v.f) and by adding weight functions.
The results concerning v.v.f comes almost for free and is proved with the same technique as the inequalities in chapter 1. The reason for considering them is that they might be useful in the context of systems of PDE (see for example chapter 4).
For most applications, the case $p = 2$ is the most interesting one, mainly because the function spaces in question then are Hilbert spaces. When considering systems of PDE between Banach(non-Hilbert) spaces the inequalities for other p should sometimes come in handy.
We also prove a Hardy inequality(HI) including higher derivatives here :

$$\int_a^b |\mathbf{V}^{(m)}(t)|^p dt \geq \sum_{k=0}^{m} \frac{A_{k,m}(p)}{(b-a)^{kp}} \int_a^b \frac{|\mathbf{V}(t)|^p}{\rho(t)^{(m-k)p}} dt, \quad \mathbf{V} \in \overset{\circ}{\mathrm{W}}{}^{1,p}([a,b])^M$$

where $\rho(t) = min\{t-a, b-t\}$,$M \in \mathbb{N}_+$ and $A_{k,m}(p)$ are positive constants and the leading constant is

$$A_{0,m}(p) = \prod_{k=1}^{m} \left(\frac{kp-1}{p}\right)^p.$$

In one dimension the leading constant is probably the best possible, but when we try to get a higher dimensional version we make some quite crude estimates, so the leading contant is most likely not the optimal one in this case.
In the third chapter we give some HI for conic domains, where we consider not just one distance function, but one for every side of the cones. One of the main results there is the inequality

$$\int_\Omega |\nabla v|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|v|^2}{|x|^2} dx + \left(\frac{2n-1}{4n^2}\right) \int_\Omega \left(\frac{1}{x_1^2} + \cdots \frac{1}{x_n^2}\right) |v|^2 dx,$$

where

$$\Omega = \{x_1 \geq 0, \ldots, x_n \geq 0\} \subset \mathbb{R}^n, \quad v \in \overset{\circ}{\mathrm{W}}{}^{1,2}(\Omega).$$

Finally, in the last chapter we give some typical applications of HI. We will show how these inequalities may be used to show existence and uniqueness of solution to a particular version of the Navier-Stokes equations in a domain Ω. The HI with remainder term obtained in chapter 1 and 2 will be used to improve some estimates when the volume of Ω is small.

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Chapter 1

A geometrical version of Hardy's inequality for $\overset{\circ}{\mathbf{W}}{}^{1,p}(\Omega)$

Abstract

The aim of this article is to prove a Hardy type inequality, concerning functions in $\overset{\circ}{W}{}^{1,p}(\Omega)$ for some domain $\Omega \subset R^n$, involving the volume of Ω and the distance to the boundary of Ω. The inequality is a generalization of a recently proved inequality by M.Hoffmann–Ostenhof, T.Hoffmann–Ostenhof and A.Laptev [13], which dealt with the special case $p = 2$.

1.1 Introduction

The history of Hardy type inequalities goes back to Hardy and the 1920's when the following original one-dimensional inequality appeared in [12].

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx,$$

where

$$p > 1, \quad f(x) \ge 0 \text{ and } F(x) = \int_0^x f(t)dt$$

(see also [11]). A multidimensional version of this inequality is

$$\int_{R^n} |\nabla u|^p dx \ge \left|\frac{n-p}{p}\right|^p \int_{R^n} \frac{|u(x)|^p}{|x|^p} dx, \quad u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}),$$

where $p > 1$ and the constant $\left|\frac{n-p}{p}\right|^p$ is optimal (see for example [20]).

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Later on, these inequalities have been generalized and modified in many different ways and the literature concerning such inequalities is extensive. There is an entire book by B.Opic and A.Kufner devoted to various Hardy type inequalities (see [20]). Many other Hardy-Sobolev type inequalities may be found in the excellent book "Sobolev Spaces" [19] by V.G.Maz'ja.
In the past few years a lot of articles on the subject has been published, see [6] for a review of recent results in the field. In the article [1] G.Barbatis, S.Filippas and A.Tertikas give a very comprehensive treatment of improved L^p Hardy inequalities with best constants, involving various kinds of distance functions.

Let Ω be an open set in $\mathbb{R}^n$ and $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. It is known (see for example [18]) that for any $p > 1$ we have

$$\int_\Omega |\nabla u|^p dx \geq c_p \int_\Omega \frac{|u|^p}{\delta(x)^p} dx, \quad u \in \overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega), \tag{1.1}$$

where Ω is convex and $c_p = \left(\frac{p-1}{p}\right)^p$ is the best constant (see for example [18]). $\overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega)$ as usual is the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm

$$||u||_{\mathrm{W}^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + \sum_{i=1}^n \Big|\Big|\frac{\partial u}{\partial x_i}\Big|\Big|_{L^p(\Omega)}.$$

The main result to be proved in this paper is that an extra term of the form

$$\frac{a(p,n)}{|\Omega|^{\frac{p}{n}}} \int_\Omega |u(x)|^p dx, \quad (a(p,n) > 0),$$

where $|\Omega|$ =vol(Ω), may be added to the right hand side of the inequality (1.1).

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary convex domain. In [3] H.Brezis and M.Marcus proved that the largest possible constant $\lambda(\Omega)$ in the inequality

$$\int_\Omega |\nabla u(x)|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\delta^2(x)} dx + \lambda(\Omega) \int_\Omega |u(x)|^2 dx, \quad u \in \overset{\circ}{\mathrm{W}}{}^{1,2}(\Omega),$$

satisfies

$$\lambda(\Omega) \geq \frac{1}{4 \cdot \mathrm{diam}^2(\Omega)}.$$

In the same paper H.Brezis and M.Marcus have asked whether the above estimate can be replaced by some other estimate of the type $\lambda(\Omega) \geq \alpha|\Omega|^{-2/n}$ for some universal constant $\alpha > 0$.
This question was recently answered affirmative by M.Hoffmann–Ostenhof, T.Hoffmann–Ostenhof and A.Laptev in [13]. In that paper, the following Hardy type inequality

$$\int_\Omega |\nabla u(x)|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\delta^2(x)} dx + \frac{\mu_n}{|\Omega|^{\frac{2}{n}}} \int_\Omega |u(x)|^2 dx, \quad u \in \overset{\circ}{\mathrm{W}}{}^{1,2}(\Omega), \tag{1.2}$$

where

$$\mu_n = \frac{n^{(n-2)/n} |\mathbb{S}^{n-1}|^{2/n}}{4},$$

is established.

Here we shall prove a similar "geometric" inequality for functions from the Sobolev space $\overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega)$. More precisely, we shall prove

$$\int_\Omega |\nabla u|^p dx \geq c_p \int_\Omega \frac{|u(x)|^p}{\delta^p(x)} dx + \frac{a(p,n)}{|\Omega|^{\frac{p}{n}}} \int_\Omega |u(x)|^p dx, \quad u \in \overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega),$$

where

$$a(p,n) = \frac{(p-1)^{p+1}}{p^p} \cdot \left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{p}{n}} \cdot \frac{\sqrt{\pi}\cdot\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}.$$

The latter inequality is a generalization of inequality (1.2) for any $p > 1$. In particular, $a(2,n) = \mu_n$.
In section one, we shall, following a method from [13], prove a one-dimensional version of the inequality and in section two, we shall extend it to higher dimensions.

1.2 One-dimensional inequalities

Let f be a function defined and differentiable on $(0, b]$ for some $b > 0$. We say that f belongs to the class $\Phi_p(0, b)$ if f is real-valued and there exists a constant $C = C(f)$ such that

$$\sup_{0<t\leq b} (t^{p-1}|f(t)| + t^p|f'(t)|) \leq C.$$

Throughout this article it is assumed that $p > 1$.

Lemma 1.1. *Let $u \in C^1[0, b], b > 0, u(0) = 0, f \in \Phi_p(0, b)$. Then we have the following inequality :*

$$\int_0^b |u'(t)|^p dt \geq \frac{1}{p^p} \frac{|\int_0^b f'(t)|u(t)|^p dt|^p}{\left(\int_0^b |(f(t) - f(b))|^{\frac{p}{p-1}} |u(t)|^p dt\right)^{p-1}}.$$

Proof. Let c be a constant. We have :

$$\begin{aligned}
|(f(b) - c)|u(b)|^p \quad &- \quad \int_0^b f'(t)|u(t)|^p dt| \\
&= \left|\int_0^b (f(t) - c)(|u(t)|^p)' dt\right| \\
&= \frac{p}{2}\left|\int_0^b (f(t) - c)(u^{\frac{p}{2}-1}\bar{u}^{\frac{p}{2}}u' + \bar{u}^{\frac{p}{2}-1}u^{\frac{p}{2}}\bar{u}')dt\right| \\
&\leq p\int_0^b |f(t) - c||u|^{p-1}|u'|dt \\
&\leq p\left(\int_0^b |(f(t) - c)|^{\frac{p}{p-1}}|u|^p dt\right)^{\frac{p-1}{p}} \left(\int_0^b |u'|^p dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Now put $c = f(b)$ and rise both sides to the power p. We get

$$\left| \int_0^b f'(t)|u|^p dt \right|^p \leq p^p \left(\int_0^b |(f(t) - f(b))|^{\frac{p}{p-1}} |u|^p dt \right)^{p-1} \int_0^b |u'|^p dt$$

and we are done. □

Corollary 1.1. *Let u be as in the lemma above and put $f(t) = \frac{t^{1-p}}{1-p} \in \Phi_p(0,b)$. Then the following improved Hardy inequality holds*

$$\begin{aligned} \int_0^b |u'(t)|^p dt &\geq c_p \frac{\left(\int_0^b \frac{|u|^p}{t^p} dt \right)^p}{\left(\int_0^b |(t^{1-p} - b^{1-p})|^{\frac{p}{p-1}} |u|^p dt \right)^{p-1}} \\ &\geq c_p \int_0^b \frac{|u|^p}{t^p} dt, \end{aligned}$$

where $c_p = \left(\frac{p-1}{p} \right)^p$.

Proof. Use the previous lemma. □

Now we give a linearized version of the corollary :

Corollary 1.2 (linearized version). *Let u be as above. Then*

$$\int_0^b |u'(t)|^p dt$$

$$\geq c_p \left(\int_0^b \left(\frac{p}{t^p} - (p-1)(t^{1-p} - b^{1-p})^{\frac{p}{p-1}} \right) |u|^p dt \right). \tag{1.3}$$

Proof. Young's inequality gives us

$$\frac{A^p}{B^{p-1}} \geq pA - (p-1)B.$$

If we put $A = \int_0^b \frac{|u|^p}{t^p} dt$, $B = \int_0^b |(t^{1-p} - b^{1-p})|^{\frac{p}{p-1}} |u|^p dt$ and use Corollary 1.1, we get (1.3).

□

An easy consequence of Corollary 1.2 (see also [13]) is

Lemma 1.2. *Let $u \in \overset{\circ}{W}{}^{1,p}(0,2b)$, $b > 0$. Then we have*

$$\int_0^{2b} |u'(t)|^p dt$$

$$\geq c_p \int_0^{2b} \left(\frac{p}{\rho(t)^p} - (p-1) \left(\frac{1}{\rho(t)^{p-1}} - \frac{1}{b^{p-1}} \right)^{\frac{p}{p-1}} \right) |u(t)|^p dt,$$

where

$$\rho(t) = \mathrm{dist}(t, \mathbb{R} \setminus [0,2b]) = \min(t, 2b-t).$$

Proof. By rewriting the inequality (1.3) for the interval $[b, 2b]$ for functions $u \in C^1[b, 2b]$ such that $u(2b) = 0$, we get

$$\int_b^{2b} |u'(t)|^p dt$$

$$\geq c_p \int_b^{2b} \left(\frac{p}{(2b-t)^p} - \frac{(p-1)}{((2b-t)^{1-p} - b^{1-p})^{\frac{-p}{p-1}}} \right) |u|^p dt. \qquad (1.4)$$

If we add the inequalities (1.3) and (1.4) and use standard density arguments, we get the statement of the lemma. $\square$

Theorem 1.1 (one-dimensional version). *Let $u \in \overset{\circ}{W}{}^{1,p}(a, b)$. Then we have*

$$\int_a^b |u'(t)|^p dt \geq c_p \left(\int_a^b \frac{|u(t)|^p}{\rho(t)^p} dt + \frac{p-1}{\left(\frac{b-a}{2}\right)^p} \int_a^b |u(t)|^p dt \right). \qquad (1.5)$$

Proof. Without loss of generality we can assume the interval of integration is $[0, 2b]$. The right hand side in Lemma 1.2 may be written

$$c_p \left(\int_0^{2b} \frac{|u(t)|^p}{\rho(t)^p} dt + \int_0^{2b} \frac{p-1}{\rho(t)^p} \left(1 - \left(1 - \left(\frac{\rho(t)}{b} \right)^{p-1} \right)^{\frac{p}{p-1}} \right) |u(t)|^p dt \right).$$

We will now estimate the expression in front of $|u|^p$ in the last integral from below. We begin by noticing that $\rho(t) \leq b$. We get :

$$\begin{aligned} \frac{1}{\rho(t)^p} \left(1 - \left(1 - \left(\frac{\rho(t)}{b} \right)^{p-1} \right)^{\frac{p}{p-1}} \right) &\geq \frac{1}{\rho(t)^p} \left(1 - \left(1 - \left(\frac{\rho(t)}{b} \right)^{p-1} \right) \right) \\ &= \frac{1}{\rho(t) b^{p-1}} \geq \frac{1}{b^p}. \end{aligned}$$

This, together with Lemma 1.2, immediately gives us inequality (1.5). $\square$

1.3 Inequalities in higher dimensions

In this section we will extend the one–dimensional results in the previous section to higher dimensions, using almost the same arguments as in [13]. For simplicity I use the same notation as in the mentioned article. If $\nu \in \mathbb{S}^{n-1}$, we put

$$\tau_\nu(x) = \min\{s > 0 : x + s\nu \notin \Omega\}, \quad \rho_\nu(x) = \min(\tau_\nu(x), \tau_{-\nu}(x))$$

$$D_\nu(x) = \tau_\nu(x) + \tau_{-\nu}(x), \quad \Omega_x = \{y \in \Omega : x + t(y - x) \in \Omega, \forall t \in [0, 1]\}$$

$$\delta(x) = \inf_{\nu \in S^{n-1}} \tau_\nu(x) = \operatorname{dist}(x, \partial\Omega)$$

$d\omega(\nu)$ denotes the normalized surface measure on $\mathbb{S}^{n-1}$, $\int_{S^{n-1}} d\omega(\nu) = 1$. Before stating our main theorem we need an auxilary lemma.

Lemma 1.3.

$$\int_{S^{n-1}} \left(\frac{2}{D_\nu(x)} \right)^p d\omega(\nu) \geq \left(\frac{n|\Omega_x|}{|\mathbb{S}^{n-1}|} \right)^{-\frac{p}{n}}. \qquad (1.6)$$

Proof. Since the function $f(t) = t^{-p}$ is convex when $t > 0$, we can use Jensens inequality to get

$$\int_{S^{n-1}} \left(\frac{2}{D_\nu(x)}\right)^p d\omega(\nu) \geq \left(\int_{S^{n-1}} \left(\frac{D_\nu(x)}{2}\right) d\omega(\nu)\right)^{-p}.$$

Since

$$\begin{aligned}
\int_{S^{n-1}} \left(\frac{D_\nu(x)}{2}\right) d\omega(\nu) &= \frac{1}{2}\int_{S^{n-1}} \tau_\nu + \tau_{-\nu} d\omega(\nu) \\
&= \int_{S^{n-1}} \tau_\nu d\omega(\nu) \\
&\leq \left(\int_{S^{n-1}} \tau_\nu^n d\omega(\nu)\right)^{\frac{1}{n}} \\
&= \left(\frac{n|\Omega_x|}{|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n}}
\end{aligned}$$

we obtain (1.6).

□

We are now ready for the main theorem.

Theorem 1.2. *Let Ω be a domain in $\mathbb{R}^n$. Then the following Hardy-type inequality holds for all $u \in \overset{\circ}{W}{}^{1,p}(\Omega)$, $p > 1$:*

$$\begin{aligned}
\int_\Omega |\nabla u|^p dx \quad &\geq \quad \frac{c_p\sqrt{\pi}\cdot\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\int_\Omega\int_{S^{n-1}} \frac{1}{\rho_v(x)^p} d\omega(\nu)|u(x)|^p dx\right. \\
&+ \quad \left.(p-1)\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{p}{n}} \int_\Omega \frac{|u(x)|^p}{|\Omega_x|^{\frac{p}{n}}} dx\right) \qquad (1.7)
\end{aligned}$$

Proof. Clearly, we can assume $u \in C_0^\infty(\Omega)$. At first, we also assume that u is real valued. E.B.Davies arguments (see [9]) together with the one-dimensional inequality (Theorem 1.1) gives

$$\int_\Omega |\partial_\nu u|^p dx \geq c_p \int_\Omega \frac{|u(x)|^p}{\rho_\nu(x)^p} dx + c_p(p-1)\int_\Omega \left(\frac{2}{D_\nu(x)}\right)^p |u(x)|^p dx.$$

By definition, we have

$$|\partial_\nu u| = |\nu \cdot \nabla u| = |\nabla u||\cos(\nu, \nabla u)|,$$

where $\cos(v, w)$ denotes the angle between $v, w \in \mathbb{R}^n$.
By inserting this into the above inequality and integrating both sides with respect to the normalized surface measure on $\mathbb{S}^{n-1}$, we get

$$\int_\Omega\int_{S^{n-1}} |\cos(\nu, \nabla u)|^p d\omega(\nu)|\nabla u(x)|^p dx \geq \qquad (1.8)$$

$$c_p\left(\int_\Omega\int_{S^{n-1}} \left(\frac{1}{\rho_v(x)^p} + (p-1)\left(\frac{2}{D_\nu(x)}\right)^p\right) d\omega(\nu)|u(x)|^p dx\right). \qquad (1.9)$$

Now note that

$$\int_{S^{n-1}} |\cos(\nu, \nabla u)|^p d\omega(\nu) = \int_{S^{n-1}} |\cos(e, \nu)|^p d\omega(\nu)$$

for any fixed unit vector $e \in \mathbb{R}^n$. Elementary calculations shows that

$$\int_{S^{n-1}} |\cos(e, \nu)|^p d\omega(\nu) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\cdot\Gamma\left(\frac{n+p}{2}\right)}.$$

By dividing both sides in (1.8), (1.9) with the latter quantity and using the above lemma, we get

$$\begin{aligned}\int_\Omega |\nabla u|^p dx \quad &\geq \quad \frac{c_p\sqrt{\pi}\cdot\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\int_\Omega\int_{S^{n-1}} \frac{1}{\rho_v(x)^p} d\omega(\nu)|u(x)|^p dx\right.\\ &+ \quad \left.(p-1)\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{p}{n}}\int_\Omega \frac{|u(x)|^p}{|\Omega_x|^{\frac{p}{n}}}dx\right)\end{aligned}$$

as desired. By standard density arguments, we get the same inequality for all real-valued $u \in \overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega)$.
Now take an arbitrary $v(x) \in C_0^\infty(\Omega)$ (not nessesarily real-valued).
Then we have $|v| \in \overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega)$. Hence, we get the inequality (1.7) for the function $u(x) = |v(x)|$.

Since $|\nabla|v(x)|| \leq |\nabla v(x)|$ a.e (see for example E.H.Lieb and M.Loss [17], p.144), we get inequality (1.7) for all $u \in C_0^\infty(\Omega)$ and thus for all $u \in \overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega)$. This concludes the proof of the theorem.

□

For convex domains an easy geometric argument shows that

$$\int_{S^{n-1}} \frac{1}{\rho_\nu(x)^p} d\omega(\nu) \geq \int_{S^{n-1}} |\cos(e,\nu)|^p d\omega(\nu)\cdot\frac{1}{\delta(x)^p} = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\cdot\Gamma\left(\frac{n+p}{2}\right)\delta(x)^p}.$$

For such domains we also know that $\Omega = \Omega_x$ for every $x \in \Omega$. Using the above theorem, we get

Theorem 1.3. *For any convex domain $\Omega \subset \mathbb{R}^n$ and $u \in \overset{\circ}{\mathrm{W}}{}^{1,p}(\Omega)$ we have*

$$\int_\Omega |\nabla u|^p dx \geq c_p \int_\Omega \frac{|u(x)|^p}{\delta^p(x)}dx + \frac{a(p,n)}{|\Omega|^{\frac{p}{n}}}\int_\Omega |u(x)|^p dx, \tag{1.10}$$

where

$$a(p,n) = \frac{(p-1)^{p+1}}{p^p}\cdot\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{p}{n}}\cdot\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}.$$

If Ω is not convex, we have the following counterpart of Corollary 3.1 in [13].

Corollary 1.3. *Suppose there exist a constant κ such that for each $y \in \partial\Omega$ and each $a > 0$ there exists a ball B with centre z disjoint from Ω and radius $\beta \geq a\kappa$, where $|z-y| = a$. Then there exists a constant $\theta \leq c_p$ such that*

$$\frac{c_p\cdot\sqrt{\pi}\cdot\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\int_{S^{n-1}}\frac{1}{\rho_\nu(x)^p}d\omega(\nu) \geq \theta\frac{1}{\delta^p(x)}$$

and hence

$$\begin{aligned}\int_\Omega |\nabla u|^p dx &\geq \theta \int_\Omega \int_{S^{n-1}} \frac{1}{\rho_v(x)^p} d\omega(\nu) |u(x)|^p dx \\ &+ (p-1) \frac{c_p \sqrt{\pi} \cdot \Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{p}{n}} \int_\Omega \frac{|u(x)|^p}{|\Omega_x|^{\frac{p}{n}}} dx.\end{aligned}$$

Chapter 2

Hardy inequalities involving higher derivatives and weights

Here we will generalize the results in the previous chapter to include higher derivatives and vectorvalued functions. It turns out that it is fairly easy to derive these inequalities, but the question of finding optimal constants is far more subtle. The reason for considering the case of vectorvalued functions is that such inequalities may be useful when one studies systems of PDEs. In chapter 3 I will give some examples of applications to the Navier-Stokes equations.

Theorem 2.1 (Generalized Hardy inequality for vectorvalued functions). *Let f be a real-valued, differentiable function on $(0, b]$ for some $b > 0$ such that*

$$\sup_{0<t\leq b} (t^{p-1}|f(t)| + t^p|f'(t)|) \leq C$$

for some constant C.
Furthermore, let

$$\mathbf{Z}(t) = (z_1(t), \ldots, z_m(t))$$

where

$$z_k(t) \in C^1[0, b] \text{ and } z_k(0) = 0, \quad k = 1, 2 \ldots, m.$$

Then we get the inequality

$$\int_0^b |\mathbf{Z}'(t)|^p dt \geq \frac{1}{p^p} \frac{\left|\int_0^b f'(t)|\mathbf{Z}(t)|^p dt\right|^p}{\left(\int_0^b |f(t) - f(b)|^{\frac{p}{p-1}} |\mathbf{Z}(t)|^p dt\right)^{p-1}}$$

Proof. Partial integration and Hölder gives us

$$\left|\int_0^b f'(t)|\mathbf{Z}(t)|^p\right| = \left|\int_0^b (f(t) - f(b)) \left(|\mathbf{Z}(t)|^p\right)' dt\right|$$

$$
\begin{aligned}
&= \left| \frac{p}{2} \int_0^b (f(t) - f(b)) |\mathbf{Z}(t)|^{p-2} \sum_{k=1}^{m} (z_k'(t)\bar{z}_k(t) + z_k(t)\bar{z}_k'(t)) dt \right| \\
&\leq p \int_0^b |f(t) - f(b)| |\mathbf{Z}(\mathbf{t})|^{p-1} |\mathbf{Z}'(t)| dt \\
&\leq p \left(\int_0^b |f(t) - f(b)|^{\frac{p}{p-1}} |\mathbf{Z}(t)|^p dt \right)^{\frac{p-1}{p}} \left(\int_0^b |\mathbf{Z}'(t)|^p \right)^{\frac{1}{p}}
\end{aligned}
$$

This gives us the desired result. □

We may use exactly the same argument as in chapter 1 to get

Theorem 2.2. *Let* $\mathbf{Z}(t) = (z_1(t), \ldots, z_m(t))$ *where* $z_k(t) \in \overset{\circ}{W}{}^{1,p}(a,b), \quad k = 1, 2, \ldots, m,$ *then*

$$
\int_a^b |\mathbf{Z}'(t)|^p dt \geq \left(\frac{p-1}{p} \right)^p \left(\int_a^b \frac{|\mathbf{Z}(\mathbf{t})|^p}{\rho(t)^p} dt + \frac{p-1}{\left(\frac{b-a}{2}\right)^p} \int_a^b |\mathbf{Z}(t)|^p dt \right) \tag{2.1}
$$

where $\rho(t)$ *as usual denotes the distance to the boundary of* (a, b).

Corollary 2.1. *Let* $\mathbf{Z}(\mathbf{t})$ *be as in the theorem and have length =* $dim(\Omega)$ *and let* $\mathbf{A}(t)$ *be a unitary matrix with* C^1 *components, then*

$$
\int_a^b |(\mathbf{A}(\mathbf{t})\mathbf{Z}(\mathbf{t}))'(t)|^p dt \geq \left(\frac{p-1}{p} \right)^p \left(\int_a^b \frac{|\mathbf{Z}(\mathbf{t})|^p}{\rho(t)^p} dt + \frac{p-1}{\left(\frac{b-a}{2}\right)^p} \int_a^b |\mathbf{Z}(t)|^p dt \right)
$$

Proof. Just apply the theorem with the vector $\mathbf{A}(\mathbf{t})\mathbf{Z}(\mathbf{t})$. □

Using the same notation and argument as in chapter 1 we get

Theorem 2.3. *Let* Ω *be an open domain in* $\mathbf{R}^n$ *and* $\mathbf{Z}(x) = (z_1(x), \ldots, z_m(x))$ *where* $z_k(x) \in \overset{\circ}{W}{}^{1,p}(\Omega), \quad k = 1, 2, \ldots, m,$ *then*

$$
\int_\Omega |\partial_\nu \mathbf{Z}(x)|^p dx \geq c_p \left(\int_\Omega \frac{|\mathbf{Z}(x)|^p}{\rho_\nu(x)^p} dx + (p-1) \int_\Omega \left(\frac{2}{D_\nu(x)} \right)^p |\mathbf{Z}(x)|^p dx \right).
$$

From now on I always assume that the domain in question is convex. If we integrate (with respect to ν) both sides of the inequality above over S^{n-1} (normalized surface measure) as in chapter 1, we may estimate the right hand side by below in exactly the same way as in chapter 1. The left hand sides is equal to

$$
\int_\Omega \int_{S^{n-1}} |(\nu \cdot \nabla z_1(x), \ldots, \nu \cdot \nabla z_m(x))|^p d\omega(\nu) dx.
$$

The inner integral is

$$
\int_{S^{n-1}} \left(|\nabla z_1|^2 |\cos(\nu, \nabla z_1)|^2 + \ldots + |\nabla z_m|^2 |\cos(\nu, \nabla z_m)|^2) \right)^{p/2} d\omega(\nu)
$$

We would like to estimate this from above by

$$
|\nabla \mathbf{Z}(x)|^p \int_{S^{n-1}} |\cos(v, e_1)|^p d\omega(\nu)
$$

where

$$|\nabla \mathbf{Z}|^p = \left(\sum_{k=1}^{m} \sum_{j=1}^{n} |\frac{\partial z_k}{\partial x_j}|^2 \right)^{\frac{p}{2}} \quad \text{and } e_1 \in S^{n-1} \text{ is an arbitrary vector .}$$

If $p \geq 2$, then we can use the convexity of $f(x) = x^{\frac{p}{2}}$ to get

$$\begin{aligned}
& \int_{S^{n-1}} \left(|\nabla z_1|^2 |\cos(\nu, \nabla z_1)|^2 + \ldots + |\nabla z_m|^2 |\cos(\nu, \nabla z_m)|^2) \right)^{p/2} d\omega(\nu) \\
\leq \quad & |\nabla \mathbf{Z}(x)|^p \int_{S^{n-1}} \frac{|\nabla z_1|^2}{\sum_{k=1}^m |\nabla z_k|^2} |\cos(\nu, \nabla z_1)|^p \\
+ \quad \ldots \quad & + \frac{|\nabla z_m|^2}{\sum_{k=1}^m |\nabla z_k|^2} |\cos(\nu, \nabla z_m)|^p) d\omega(\nu) \\
= \quad & |\nabla \mathbf{Z}(x)|^p \int_{S^{n-1}} |\cos(v, e_1)|^p d\omega(\nu).
\end{aligned}$$

Altogether, we get the inequality

$$\int_\Omega |\nabla \mathbf{Z}(x)|^p dx \geq \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{|\mathbf{Z}(x)|^p}{\rho(x)^p} dx + \frac{a(p,n)}{|\Omega|^{\frac{p}{n}}} \int_\Omega |\mathbf{Z}(x)|^p dx.$$

where $a(p,n)$ is the same constant as in chapter 1. Note that the length of the vector $\mathbf{Z}(x)$ is arbitrary.

This proof does not work if $1 \leq p < 2$ ($f(x) = x^{\frac{p}{2}}$ because fails to be convex). In [21] P.E.Sobolevskii proves this inequality for all $p \geq 1$, but without the remainder term and where the length of the vectors equals the dimension of Ω. I therefore suspect that it might be possible to prove the above theorem also when $1 \leq p < 2$.

Corollary 2.2. *Let $u \in C_0^\infty(\Omega)$ $2 \leq p < \infty$ (Ω convex) then*

$$\int_\Omega |\nabla_2 u| \geq \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{|\nabla u|^p}{\rho(x)^p} dx + \frac{a(p,n)}{|\Omega|^{\frac{p}{n}}} \int_\Omega |\nabla u|^p. \tag{2.2}$$

Proof. Just put $Z = \nabla u$ in the previous theorem. □

We shall now generalize the above theorem and the ones in chapter one to higher derivatives.

Theorem 2.4. *Let $\mathbf{U}(\mathbf{t})$ be a vector of arbitrary length m, with components $u_k(t) \in \overset{\circ}{W}{}^{n,p}(a,b)$. Furthermore, let $d \in [0, np-1)$ be parameter. Then we have the following generalization of theorem (1.1).*

$$\begin{aligned}
\int_a^b \frac{|\mathbf{U}'(t)|^p}{\rho(t)^{(n-1)p-d}} dt \quad & \geq \quad \left(\frac{np-d-1}{p} \right)^p \left(\int_a^b \frac{|\mathbf{U}(t)|^p}{\rho(t)^{np-d}} dt \right. \\
& + \quad \left. \frac{p-1}{\left(\frac{b-a}{2}\right)^{np-d}} \int_a^b |\mathbf{U}(t)|^p dt \right).
\end{aligned}$$

Proof. We may assume that the interval in question is $[0,2b]$.
Let $u_k(t) \in C^n[0,b]$ where
$u_k(0) = u_k'(0) = \ldots = u_k^{(n-1)}(0) = 0, \quad k = 1, \ldots, m.$
Partial integration gives (note that the boundary terms vanishes due to the boundary conditions on $\mathbf{U}$)

$$\begin{aligned}
& \int_0^b \frac{|\mathbf{U}(t)|^p}{t^{np-a}} dt \\
\leq \; & \frac{1}{np-a-1} \int_0^b \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right) \left(|\mathbf{U}(t)|^p \right)' |dt \\
\leq \; & \frac{p}{np-a-1} \int_0^b \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right) |\mathbf{U}(t)|^{p-1} |\mathbf{U}'(t)| dt \\
= \; & \frac{p}{np-a-1} \int_0^b \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right)^{\frac{p-1}{p}} t^{-\frac{p-1}{p}} |\mathbf{U}(t)|^{p-1} \\
\cdot \; & \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right)^{\frac{1}{p}} t^{\frac{p-1}{p}} |\mathbf{U}'(t)| dt \\
\leq \; & \frac{p}{np-a-1} \left(\int_0^b \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right) \frac{1}{t} |\mathbf{U}(t)|^p dt \right)^{\frac{p-1}{p}} \\
\cdot \; & \left(\int_0^b \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right) t^{p-1} |\mathbf{U}'(t)|^p dt \right)^{\frac{1}{p}},
\end{aligned}$$

hence we have

$$\int_0^b \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right) t^{p-1} |\mathbf{U}'(t)|^p dt \geq$$

$$\geq \left(\frac{np-a-1}{p} \right)^p \frac{\left(\int_0^b \frac{|\mathbf{U}(t)|^p}{t^{np-a}} dt \right)^p}{\left(\int_0^b \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right) \frac{1}{t} |\mathbf{U}(t)|^p dt \right)^{p-1}}$$

The inequality

$$\frac{A^p}{B^{p-1}} \geq pA - (p-1)B, \quad A, B \geq 0$$

gives

$$\begin{aligned}
& \int_0^b \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right) t^{p-1} |\mathbf{U}'(t)|^p dt \\
\geq \; & D(a,p,n) \int_0^b \frac{|\mathbf{U}(t)|^p}{t^{np-a}} dt + \\
& +(p-1) D(a,p,n) \int_0^b \left(\frac{1}{t^{np-a}} - \left(\frac{1}{t^{np-a-1}} - \frac{1}{b^{np-a-1}} \right) \frac{1}{t} \right) |\mathbf{U}(t)|^p dt, \quad (2.3)
\end{aligned}$$

where

$$D(a,p,n) = \left(\frac{np-a-1}{p} \right)^p$$

A crude estimate gives us

$$\int_0^b \frac{|\mathbf{U}'(t)|^p}{t^{(n-1)p-a}} \geq \left(\frac{np-a-1}{p}\right)^p \left(\int_0^b \frac{|\mathbf{U}(t)|^p}{t^{np-a}}dt + \frac{p-1}{b^{np-a}}\int_0^b |\mathbf{U}(t)|^p dt\right).$$

The same argument as in chapter 1 leads to

$$\begin{aligned}\int_a^b \frac{|\mathbf{U}'(t)|^p}{\rho(t)^{(n-1)p-a}} &\geq \left(\frac{np-a-1}{p}\right)^p \left(\int_a^b \frac{|\mathbf{U}(t)|^p}{\rho(t)^{np-a}}dt\right.\\ &+ \left.\frac{p-1}{\left(\frac{b-a}{2}\right)^{np-a}}\int_a^b |\mathbf{U}(t)|^p dt\right)\end{aligned}$$

valid for every $\mathbf{U}(t)$ with components in $\overset{\circ}{W}{}^{n,p}(a,b)$. □

Remark : From inequality (2.3) we get

$$\begin{aligned}&\int_0^b \frac{|\mathbf{U}'(t)|^p}{t^{(n-1)p-a}}dt - \frac{1}{b^{np-1}}\int_0^b t^{p+a-1}|\mathbf{U}'(t)|^p dt \geq \\ &D(a,p,n)\int_0^b \frac{|\mathbf{U}(t)|^p}{t^{np-a}}dt + \frac{(p-1)D(a,p,n)}{b^{np-1}}\int_0^b t^{a-1}|\mathbf{U}(t)|^p dt.\end{aligned}$$

This inequality was independently obtained by F. Colin and Y. Hupperts [5] in the case $n = 1, p = 2$ and U is a scalar valued function compactly supported in $(0, b)$. They used it along with other lemmas to prove some results concerning weighted Hardy inequalities in higher dimensions.

Remark 2: If we put $n = 1, d = 0$ we get theorem 1.1 as an immediate corollary.

The above theorem can be used to generalize the Hardy inequality in the previous chapter to higher derivatives.

Corollary 2.3. *Let $\mathbf{V}(t)$ be a vector of arbitrary length with components in $\overset{\circ}{W}{}^{m,p}(a,b)$. Then*

$$\int_a^b |\mathbf{V}^{(m)}(t)|^p dt \geq \sum_{k=0}^{m} \frac{A_{k,m}(p)}{(b-a)^{kp}}\int_a^b \frac{|\mathbf{V}(t)|^p}{\rho(t)^{(m-k)p}}dt, \tag{2.4}$$

where $A_{k,m}(p)$ are positive constants and the leading constant is

$$A_{0,m}(p) = \prod_{k=1}^{m}\left(\frac{kp-1}{p}\right)^p$$

Proof. Estimate the left hand side of (2.4) by below by using the above theorem with $d = 0, m = 1$ and $U(t) = V^{(m-1)}(t)$ to get

$$\int_a^b |\mathbf{V}^{(m)}(t)|^p dt \geq c_p\left(\int_a^b \frac{|\mathbf{V}^{(m-1)}(t)|^p}{\rho(t)^p}dt + \frac{p-1}{\left(\frac{b-a}{2}\right)^p}\int_a^b |\mathbf{V}^{(m-1)}(t)|^p dt\right).$$

Now we make repeated use of the theorem to estimate the two terms in the right hand side until we get an expression without any derivatives of V involved. We arrive at an expression of the form (2.4). $\square$

As an example, in the case $m = 2$ we get

$$\begin{aligned}&\int_a^b |\mathbf{U}''(t)|^p dt \\ \geq\ & D_1(p) \int_a^b \frac{|\mathbf{U}(t)|^p}{\rho(t)^{2p}} dt + \frac{D_2(p)}{\left(\frac{b-a}{2}\right)^p} \int_a^b \frac{|\mathbf{U}(t)|^p}{\rho(t)^p} dt + \frac{D_3(p)}{\left(\frac{b-a}{2}\right)^{2p}} \int_a^b |\mathbf{U}(t)|^p dt,\end{aligned}$$

where

$$\begin{aligned}D_1(p) &= \left(\frac{p-1}{p}\right)^p \left(\frac{2p-1}{p}\right)^p, \quad D_2(p) = \left(\frac{p-1}{p}\right)^{2p} (p-1) \text{ and} \\ D_3 &= \left(\frac{p-1}{p}\right)^p \left(\frac{2p-1}{p}\right)^p (p-1) + \left(\frac{p-1}{p}\right)^{2p} (p-1)^2.\end{aligned}$$

An open question here is the problem of finding the optimal sequence of constants $\{A_k(p)\}_0^n$ in the corollary above. Optimal here means that we want an optimal $A_0(p)$ and with this constant given, we seek an optimal $A_1(p)$ and so on.
As far as I know, the problem is still unsettled even in the case $n = 1$. In this case the leading optimal constant is well known, however, and equals $(p-1)^p/p^p$, but the second constant is unknown.

Theorem 2.5. *Let $\mathbf{U}(x)$ be a vector with components in $\overset{\circ}{W}{}^{m,p}(\Omega)$, where Ω is an open domain in $\mathbb{R}^n$ then*

$$\begin{aligned}&\int_\Omega |\nabla_m \mathbf{U}|^p dx \geq A_{0,m}(p) \int_\Omega \int_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu(x)^{mp}} d\omega |\mathbf{U}(x)|^p dx \\ +\ & \left(\sum_{k=1}^m \frac{A_{k,m}(p)}{2^{kp}}\right) \left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{mp}{n}} \int_\Omega \frac{|\mathbf{U}(x)|^p}{|\Omega_x|^{\frac{mp}{n}}} dx,\end{aligned}$$

where

$$|\nabla_m \mathbf{U}|^p = \left(\sum_{k=1}^M \sum_{|\alpha|=m} |D^\alpha u_k|^2\right)^{\frac{p}{2}}, \quad \mathbf{U} = (u_1, \ldots, u_M).$$

Proof. If we apply our usual method to "translate" the inequality in the theorem to higher dimensions, we get

$$\begin{aligned}&\int_\Omega \int_{S^{n-1}} |\partial_\nu^m \mathbf{U}(x)|^p \, d\omega(\nu) dx \\ \geq\ & \sum_{k=0}^m \frac{A_{k,m}(p)}{2^{kp}} \int_\Omega \int_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(x)}\right)^{kp} \frac{1}{\rho_\nu(x)^{(m-k)p}} d\omega |\mathbf{U}(x)|^p dx\end{aligned}$$

where ∂_ν^2 is the m:th order directional derivative in the direction ν.

By lemma 1.6 we have

$$\int_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(x)}\right)^{kp} \frac{1}{\rho(x)^{(m-k)p}} d\omega \geq \int_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(x)}\right)^{mp} d\omega \geq \left(\frac{n|\Omega_x|}{|\mathbb{S}^{n-1}|}\right)^{-\frac{mp}{n}}.$$

By Cauchy-Schwarz inequality we have the estimate

$$|\partial_\nu^m \mathbf{U}(x)|^p \leq |\nabla_m \mathbf{U}|^p (v_1^2 + \cdots v_n^2)^{\frac{mp}{2}} = |\nabla_m \mathbf{U}|^p.$$

Hence we arrive at 2.5. □

As before the inequality becomes more pleasant when Ω is convex :

Corollary 2.4. *Let U be as above and let Ω be convex domain, then*

$$\int_\Omega |\nabla_m \mathbf{U}|^p dx \geq A_{0,m}(p) \frac{\Gamma\left(\frac{mp+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\cdot\Gamma\left(\frac{n+mp}{2}\right)} \int_\Omega \frac{|\mathbf{U}(x)|^p}{\delta(x)^{mp}} dx$$
$$+ \left(\sum_{k=1}^m \frac{A_{k,m}(p)}{2^{kp}}\right)\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{mp}{n}} \int_\Omega |\mathbf{U}(x)|^p dx.$$

The constant appearing in front of the integral in the leading term on the RH-side above is not the best possible. In the case $m = 1$ we know from [21] that $(p-1)^p/p^p$ is optimal. The coefficient in front of the second integral is as far as I know still not known for any m in any dimension. However, some partial results on in this issue has been obtained. In a paper by G.Barbatis, S.Filippas and A.Tertikas (see [2]) they show that when Ω is the unit ball in $\mathbb{R}^n$, B, $p = 2$ and u scalar valued we have

$$C_n := \inf_{u \in \overset{\circ}{W}{}^{1,2}(B)} \frac{\int_B |\nabla u|^2 dx - \frac{1}{4}\int_B \frac{|u|^2}{\rho(x)^2} dx}{\int_B |u|^2 dx} \geq \mu_2 + \frac{(n-1)(n-3)}{4},$$

where $\mu_2 \approx 5.783$ is the first eigenvalue of the Dirichlet Laplacian for the unit disk in $\mathbb{R}^2$. They also proved that when $n = 3$, $C_n = \mu_2$.
For arbitrary m one might expect $A_{0,m}(p)$ to be the best constant. If that is the case, then it cannot be proved by refining the estimate of integral

$$\int_{S^{n-1}} |\partial_\nu^m \mathbf{U}(x)|^p \, d\omega(\nu)$$

appearing in the proof of the theorem which the following counterexample shows :
Let the dimension $n = 2$, $m = 2$ and let $\mathbf{U}(x) = u(x)$ be scalar valued.
Now suppose $x = x_0 \in \Omega$ is such that

$$\frac{\partial^2 u(x_0)}{\partial x_1 \partial x_1} = \frac{\partial^2 u(x_0)}{\partial x_2 \partial x_2} = 1, \quad \frac{\partial^2 u(x_0)}{\partial x_1 \partial x_2} = 0.$$

Then

$$\int_{S^1} |\partial_v^2 u(x_0)|^2 d\omega(\nu) = \int_{S^1} (v_1 + v_2)^2 d\omega(\nu) |\nabla_2 u(x_0)|^2 = |\nabla_2 u(x_0)|^2,$$

but $|\nabla_2 u(x_0)|^2$ is greater than

$$\frac{\Gamma\left(\frac{5}{2}\right)}{\sqrt{\pi}\Gamma(3)} |\nabla_2 u(x_0)|^2 = \frac{3}{4}.$$

Chapter 3

Hardy inequalities in conic domains

In the previous chapters we have considered inequalities in convex domains where we were able to add remainder terms when the domain in question was bounded. Here we will consider inequalities in conic unbounded domains. In this case it is natural to consider two distance functions instead of one as before, namely the distances to each of the two sides of the cone. In [19] Mazya consider the special case where the cone is a half space and obtain the following inequality

$$\int_{\mathbb{R}^n_+} (\nabla u)^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} + \frac{1}{16} \int_{\mathbb{R}^n_+} \frac{u^2}{(x_{n-1}^2 + x_n^2)^{1/2} |x_n|} dx,$$

where $u \in C_0^\infty(\mathbb{R}^n_+)$. Here we will use analogous methods to get inequalities for arbitrary cones.

Theorem 3.1. *Let $u \in C^\infty(\Omega)$, where Ω is the first quadrant of $\mathbb{R}^2$ and let u be equal to 0 near the origin and for large $|x|$. Let a be a parameter not equal to 2. Then we have*

$$\int_\Omega \frac{|u|^p}{(x_1^2 + x_2^2)^{\frac{a}{2}}} dx \leq C_1(a,p) \int_\Omega |\nabla u|^p (x_1^2 + x_2^2)^{\frac{p-a}{2}} dx, \tag{3.1}$$

where

$$C_1(a,p) = \left(\frac{p}{|2-a|} \right)^{\frac{p}{p-1}}.$$

Proof. We use polar coordinates and integrate by parts to get

$$\begin{aligned} \int_\Omega \frac{|u|^p}{(x_1^2 + x_2^2)^{\frac{a}{2}}} dx &= \int_\theta^{\frac{\pi}{2}-\theta} \int_0^\infty |u|^p r^{1-a} dr d\theta \\ &\leq \frac{p}{|2-a|} \int_\theta^{\frac{\pi}{2}-\theta} \int_0^\infty |\frac{\partial u}{\partial r}| |u|^{p-1} r^{2-a} dr d\theta \end{aligned}$$

$$\leq \frac{p}{|2-a|}\left(\int_{\theta}^{\frac{\pi}{2}-\theta}\int_0^\infty |u|^p \frac{r}{r^a} dr d\theta\right)^{\frac{1}{p}}\left(\int_{\theta}^{\frac{\pi}{2}-\theta}\int_0^\infty |\frac{\partial u}{\partial r}|^p r^{p-a+1} dr d\theta\right)^{\frac{p-1}{p}}$$

$$\leq \frac{p}{|2-a|}\left(\int_\Omega \frac{|u|^p}{(x_1^2+x_2^2)^{\frac{a}{2}}}dx\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla u|^p (x_1^2+x_2^2)^{\frac{p-a}{2}}\right)^{\frac{p-1}{p}}.$$

Now rise both sides to the power $\frac{p}{p-1}$ to get inequality 3.1. □

Corollary 3.1. *Let $u \in C_0^\infty(\Omega)$, where $\Omega = \Omega(\theta)$ is the cone in the first quadrant of $\mathbb{R}^2$ with vertex in the origin and where the angle between the first (second) boundary line of $\Omega(\theta)$ and the y-axis (x-axis) is equal to θ. We then have the inequality*

$$\int_\Omega \frac{|u|^p}{(x_1^2+x_2^2)^{\frac{a}{2}}}dx \leq C_2(\theta)\int_\Omega |\nabla u|^p (x_1x_2)^{\frac{p-a}{2}}dx.$$

Proof. Use inequality 3.1 and observe that $x_1^2+x_2^2 \leq 2x_1x_2\tan\theta$ in $\Omega(\theta)$ □

Theorem 3.2. *Let u be as in the last corollary. Then we have the following inequality*

$$C_2(\theta)\int_\Omega \frac{|u|^2}{(x_1^2+x_2^2)^{\frac{1}{2}}(x_1x_2)^{\frac{1}{2}}}dx + \frac{3}{16}\int_\Omega\left(\frac{1}{x_1^2}+\frac{1}{x_2^2}\right)u(x)^2 dx \leq \int_\Omega |\nabla u|^2 dx. \tag{3.2}$$

where

$$C_2(\theta) = \frac{1}{4\sqrt{2\tan\theta}}.$$

Proof. Put $a=1$ and $p=2$ in the last corollary and substitute $u(x) = (x_1x_2)^{-1/4}v(x)$ to get

$$C_2(\theta)\int_\Omega \frac{|v|^2}{(x_1^2+x_2^2)^{\frac{1}{2}}(x_1x_2)^{\frac{1}{2}}}dx$$

$$\leq \int_\Omega \frac{1}{16}\left(\frac{1}{x_1^2}+\frac{1}{x_2^2}\right)v(x)^2 + |\nabla v|^2 - \frac{1}{2}\frac{1}{x_1}v\frac{\partial v}{\partial x_1} - \frac{1}{2}\frac{1}{x_2}v\frac{\partial v}{\partial x_2}dx.$$

Now integrate by parts to get (3.2) □

It would be interesting to know if it is possible to get the same inequality as in the corollary in the quarter plane, but with a positive contant instead of $C_2(\pi/2) = 0$.

Theorem 3.3. *Let v be compactly supported in the set*

$$\Omega = \{x_1 \geq 0, \ldots, x_n \geq 0\} \subset \mathbb{R}^n.$$

Then we have

$$\frac{1}{4}\int_\Omega \frac{|v|^2}{|x|^2}dx + \left(\frac{2n-1}{4n^2}\right)\int_\Omega\left(\frac{1}{x_1^2}+\cdots\frac{1}{x_n^2}\right)|v|^2 dx \leq \int_\Omega |\nabla v|^2 dx \tag{3.3}$$

Proof. By using polar coordinates we get

$$\int_\Omega \frac{|v|^2(x_1 \cdot \ldots \cdot x_n)^{\frac{1}{n}}}{x_1^2+\ldots+x_n^2}dx = \int_0^{\frac{\pi}{2}}\int_0^\infty |v|^2 f(\theta)drd\theta$$

where $f(\theta)$ doesn't depend on r. We integrate by parts with respect to r and use Hölder to get that this is not greater than

$$2\left(\int_0^{\frac{\pi}{2}}\int_0^\infty |v|^2 f(\theta)drd\theta\right)^{\frac{1}{2}}\left(\int_0^{\frac{\pi}{2}}\int_0^\infty |\nabla v|^2 r^2 f(\theta)drd\theta\right)^{\frac{1}{2}}.$$

If we square both sides of this inequality and change back to rectangular coordinates we get

$$\frac{1}{4}\int_\Omega \frac{|v|^2(x_1 \cdot \ldots \cdot x_n)^{\frac{1}{n}}}{x_1^2+\ldots+x_n^2}dx \le \int_\Omega |\nabla v|^2 (x_1 \cdot \ldots \cdot x_n)^{\frac{1}{n}}$$

Now substitute $v=(x_1 \cdot \ldots \cdot x_n)^{\frac{-1}{2n}}u$ to get

$$\frac{1}{4}\int_\Omega \frac{|u|^2}{x_1^2+\ldots+x_n^2}dx \le \int_\Omega |\nabla u|^2 dx + \frac{1}{4n^2}\int_\Omega\left(\frac{1}{x_1^2}+\ldots+\frac{1}{x_n^2}\right)|v|^2dx$$
$$-\frac{1}{2n}\sum_{k=1}^n\int_\Omega \frac{1}{x_k}\cdot\frac{\partial v^2}{\partial x_k}dx.$$

Now integrate by parts to get (3.3). □

Remark : The ordinary Hardy inequality tells us that

$$\frac{1}{4}\int_\Omega \frac{|u|^2}{dist(x,\partial\Omega)^2}dx \le \int_\Omega |\nabla u|^2 dx, \tag{3.4}$$

where $1/4$ is the optimal constant. We may note that inequality (3.3) leads to (3.4), but with a better constant, when u is supported near the diagonal $x_1 = \ldots = x_n$.

Corollary 3.2. *We also immediately get another version of the Hardy inequality*

$$C_3\int_\Omega \frac{|u|^2}{|x|^2}dx \le \int_\Omega |\nabla u|^2dx,$$

where

$$C_3 = \frac{n}{2}$$

Proof. The inequalities between harmonic, geometric and aritmetric mean values gives us

$$\frac{n}{\frac{1}{x_1^2}+\ldots+\frac{1}{x_n^2}} \le (x_1 \cdot \ldots \cdot x_n)^{\frac{2}{n}} \le \frac{x_1^2+\ldots+x_n^2}{n}.$$

Apply these inequalities to (3.2) and we are done. □

We may generalize inequality (3.3) to general conic domains. Here we consider the case $n = 2$.

Theorem 3.4. *Let $u \in C_0^\infty(\Omega(\theta))$, where $\Omega(\theta)$ is a cone with vertex in the origin and opening angle equal to $\theta \in [0, \frac{\pi}{2}]$. Then we get*

$$\frac{1}{4}\int_\Omega \frac{|v|^2}{|x|^2}dx + C_4(\theta)\int_\Omega \left(\frac{1}{\rho_1(x)^2} + \frac{1}{\rho_2(x)^2}\right)|v(x)|^2 dx \leq \int_\Omega |\nabla v|^2 dx,$$

where

$$C_4(\theta) = \frac{3}{16}\left(1 + \frac{1}{3}(1-\sin\theta)\cdot\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right).$$

and $\rho_1(x)$ and $\rho_2(x)$ are the distances to the two boundary lines of Ω

Proof. At first, we suppose the cone lies in the first quadrant and is ce ntred symmetrical around the line $y = x$. Define k_1 and $1/k_1$ to be the slopes of the two boundary lines of the cone, where $k_1 = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) > 1$.
We begin by noticing that the explicit expressions for ρ_1 and ρ_2 are

$$\rho_1(x) = \frac{k_1x_1 - x_2}{\sqrt{k_1^2+1}}, \qquad \rho_2(x) = \frac{k_1x_2 - x_1}{\sqrt{k_1^2+1}},$$

which means that in polar coordinates we have that $(\rho_1\rho_2)^{\frac{1}{2}} = rf(\theta)$ where $f(\theta)$ doesn't depend on θ. We get

$$\int_\Omega \frac{|u|^2(\rho_1\rho_2)^{\frac{1}{2}}}{|x|^2}dx = \int\int |u|^2 f(\theta)drd\theta$$

$$\begin{aligned}
&\leq 2\int\int |u||\frac{\partial u}{\partial r}|rf(\theta)drd\theta \\
&\leq 2\left(\int\int |u|^2 f(\theta)drd\theta\right)^{\frac{1}{2}}\left(\int\int |\frac{\partial u}{\partial r}|^2 r^2 f(\theta)drd\theta\right)^{\frac{1}{2}} \\
&\leq 2\left(\int_\Omega \frac{|u|^2(\rho_1\rho_2)^{\frac{1}{2}}}{|x|^2}dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^2(\rho_1\rho_2)^{\frac{1}{2}}dx\right)^{\frac{1}{2}}.
\end{aligned}$$

If we square both sides we get

$$\frac{1}{4}\int_\Omega \frac{|u|^2(\rho_1\rho_2)^{\frac{1}{2}}}{|x|^2}dx \leq \int_\Omega |\nabla u|^2(\rho_1\rho_2)^{\frac{1}{2}}dx.$$

Now substitute

$$u(x) = (\rho_1(x)\rho_2(x))^{-\frac{1}{4}}v(x) = \left(\frac{(k_1x_1 - x_2)(k_1x_2 - x_1)}{k_1^2+1}\right)v(x)$$

in the above inequality. We get

$$\begin{aligned}
0 &\leq \int_\Omega |\nabla v|^2 dx \\
&+ \frac{1}{16}\int_\Omega \frac{(-2k_1x_1 + (k_1^2+1)x_2)^2 + (-2k_1x_2 + (k_1^2+1)x_1)^2}{(-k_1(x_1^2+x_2^2) + (k_1^2+1)x_1x_2)^2}v(x)^2 dx
\end{aligned}$$

$$+\int_\Omega \left(\frac{\partial}{\partial x_1}\left((-k_1(x_1^2+x_2^2)+(k_1^2+1)x_1x_2)^{-1}(-2k_1x_1+(k_1^2+1)x_2)\right)\right)v(x)^2dx$$
$$+\int_\Omega v(x)^2\frac{\partial}{\partial x_2}\left((-k_1(x_1^2+x_2^2)+(k_1^2+1)x_1x_2)^{-1}(-2k_1x_2+(k_1^2+1)x_1)\right)dx.$$

Partial integration gives (after some simplifications)

$$0\le\int_\Omega|\nabla v|^2dx+\frac{1}{16}\int_\Omega\frac{(-3k_1^4-2k_1^2-3)(x_1^2+x_2^2)+8k_1(k_1^2+1)x_1x_2}{(-k_1(x_1^2+x_2^2)+(k_1^2+1)x_1x_2)^2}v(x)^2dx$$

Since $2x_1x_2\le x_1^2+x_2^2$ this is not greater than

$$\int_\Omega|\nabla v|^2dx+C_{k_1}\int_\Omega\frac{|x|^2v(x)^2}{\left(\rho_1(x)\rho_2(x)\right)^2}dx$$

where

$$C_{k_1}=\frac{1}{16}\left(\frac{-(k_1-1)^2(3k_1^2+2k_1+3)}{k_1^4+2k_1^2+1}\right)\le 0.$$

We would now like to have an estimate

$$\frac{C_{k_1}|x|^2}{\rho_1(x)^2\rho_2(x)^2}\le D_{k_1}\left(\frac{1}{\rho_1(x)^2}+\frac{1}{\rho_2(x)^2}\right)\Leftrightarrow\frac{C_{k_1}|x|^2}{\rho_1(x)^2+\rho_2(x)^2}\le D_{k_1}$$

for some negative constant D_1. Now put $t=y/x$. We have

$$\frac{C_{k_1}|x|^2}{\rho_1(x)^2+\rho_2(x)^2}=\frac{C_{k_1}(k_1^2+1)(x_1^2+x_2^2)}{(k_1x_1-x_2)^2+(k_1x_2-x_1)^2}=\frac{C_{k_1}(k_1^2+1)(t^2+1)}{(k_1-t)^2+(k_1t-1)^2}$$

By elementary calculus we find that the maximum value of this expression is attained when $t=1$ and equals

$$D_{k_1}:=\frac{C_{k_1}(k_1^2+1)}{(k_1-1)^2}.$$

We get the inequality

$$\frac{1}{4}\int_\Omega\frac{|v|^2}{|x|^2}dx-D_{k_1}\int_\Omega\left(\frac{1}{\rho_1(x)^2}+\frac{1}{\rho_2(x)^2}\right)|v|^2dx\le\int_\Omega|\nabla v|^2dx.$$

where

$$-D_{k_1}=\frac{3}{16}\left(1+\frac{1}{3}\left(1-\sin\theta\right)\cdot\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right).$$

To get the same inequality for any conic domain (with vertex in the origin) we just rotate the cone by a linear change of variables. □

Note that when the cone in the above theorem is equal to a quarter plane, we get inequality (3.3) with $n=2$. One may wonder what the exact values of the best constants in this inequality are. I have not found any indications of this in the litterature. These Hardy type inequalities can be generalized in many possible ways. It is not hard to generalize them to cones in higher dimensions, but the calculations becomes more involved. A more interesting generalization would be to consider the case of the corresponding L^p inequalities. As an example one may ask whether the inequality

$$\int_\Omega|\nabla u|^pdx\ge C\int_\Omega\left(\frac{1}{x_1^p}+\frac{1}{x_2^p}\right)|u|^pdx$$

holds for some positive C.

Chapter 4

Applications to the Navier-Stokes equations

4.1 Some norm estimates using Hardy's inequality

The stationary Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^3$ are

$$\begin{cases} -\nu\Delta\mathbf{v} + \sum_{k=1}^{n} v_k \frac{\partial \mathbf{v}}{\partial x_k} & = -\nabla p + \mathbf{f}(x) \\ \text{div } \mathbf{v} & = 0 \end{cases} \tag{4.1}$$

where $\mathbf{f}(\mathbf{x})$ is a known vector valued function and we seek $\mathbf{v}(\mathbf{x})$ and $p(x)$. We will here study the case when $\mathbf{v}$ restricted to $\partial\Omega$ is zero.
Following the notation from [16], we also demand that $\mathbf{v}$ should belong to the function space $\mathbf{H}(\Omega)$, which will be defined below.
Let

$$\dot{\mathbf{J}} = \{\mathbf{v} = (v_1, v_2, v_3) : \text{ div } \mathbf{v} = 0, v_k \in C_0^\infty(\Omega), k = 1, 2, 3\}$$

and let this space be equipped with the scalar product

$$(\mathbf{u}, \mathbf{v})_1 = \sum_{k=1}^{3} \int_\Omega \frac{\partial \mathbf{u}}{\partial x_k} \cdot \frac{\partial \mathbf{v}}{\partial x_k} dx$$

The (Hilbert) space $\mathbf{H}(\Omega)$ is now defined as the closure of $\dot{\mathbf{J}}$ with respect to this metric. If we take the product of both sides of the first equation in (4.1) with an arbitrary $\mathbf{w} \in \dot{\mathbf{J}}$ and integrate be parts we get the *a priore* identity

$$\sum_{k=1}^{3} \int_\Omega \nu \frac{\partial \mathbf{v}}{\partial x_k} \cdot \frac{\partial \mathbf{w}}{\partial x_k} + v_k \frac{\partial \mathbf{v}}{\partial x_k} \cdot \mathbf{w} = \int_\Omega \mathbf{f} \cdot \mathbf{w} dx \tag{4.2}$$

We say that $\mathbf{v} \in \mathbf{H}(\Omega)$ is a generalized solution to (4.1) if it satisfies equation (4.2).
Note that (4.2) does not depend on the unknown funtion p. In fact, we shall not be interested in this part of the solution in this article.
There are two key theorems (see [16]) which allow one to verify whether there exist a (generalized) solution or not and if the solution is unique.

Theorem 4.1. *Let Ω be bounded. Then the equations (4.1) (including the condition $\mathbf{v}|_{\partial\Omega} = 0$) have at least one generalized solution if $\mathbf{f}$ is such that $\int_\Omega \mathbf{f} \cdot \mathbf{w} dx$ defines a linear functional of $\mathbf{w} \in \mathbf{H}(\Omega)$ If Ω is unbounded, we only consider those $\mathbf{v}$ which in addition tend to zero when $|x| \to \infty$.*

Theorem 4.2. *Let Ω be bounded. Then we cannot have more than one solution to the problem described in the last theorem if*

$$\frac{2\sqrt{3}}{\lambda_1^{\frac{1}{4}} \nu^2} \sup_{\mathbf{w} \in \mathbf{H}(\Omega), \mathbf{w} \neq 0} \frac{\left| \int_\Omega \mathbf{f} \cdot \mathbf{w} dx \right|}{(\mathbf{w}, \mathbf{w})_1} < 1 \tag{4.3}$$

where λ_1 is the smallest eigenvalue of the Dirichlet laplacian $-\Delta$ on Ω.

In both of these theorems, Hardy inequalities are useful when one wants to prove that the conditions of the theorems are fulfiled.

By estmating the LH-side of (4.3) by using the Cauchy-Schwartz and Friedrichs inequalities we get

$$|||f||| := \sup_{\mathbf{w} \in \mathbf{H}(\Omega), \mathbf{w} \neq 0} \frac{\left| \int_\Omega \mathbf{f} \cdot \mathbf{w} dx \right|}{(\mathbf{w}, \mathbf{w})_1} \leq C ||\mathbf{f}||_{L^2(\Omega)}$$

Where the optimal constant C depends on the constant in (4.3) and on the best constant in Friedrichs inequality.
We may note that if the L^2 norm of $\mathbf{f}$ is sufficently small, the solution to the stationary Navier-stokes equations is unique.

Henceforth we assume that Ω is convex, but the examples here also hold in more general domains. The only difference will be that the best Hardy constant

$$H(\Omega) := \inf_{\mathbf{w} \in \mathbf{H}(\boldsymbol{\Omega}), \mathbf{w} \neq 0} \frac{\int_\Omega |\nabla \mathbf{w}|^2 dx}{\int_\Omega \frac{|w|^2}{\delta^2} dx}, \qquad \rho(x) = dist(x, \partial\Omega)$$

will then depend on Ω. Of course it might happen that this constant is zero for some Ω. Such Ω should then be excluded here. One can show that the constant is always non-zero if the boundary of Ω is sufficently nice, for example C^2.

If we in our estimate of $|||\mathbf{f}|||$ above use Hardy's inequality instead of Friedrichs ineq., as in [21], we get the estimate

$$|||f||| \leq \sup_{\mathbf{w} \in \mathbf{H}(\Omega), \mathbf{w} \neq 0} \frac{||\delta \mathbf{f}||_{L^2} ||\frac{\mathbf{w}}{\delta}||_{L^2}}{(\mathbf{w}, \mathbf{w})_1} \leq 2 ||\delta \mathbf{f}||_{L^2} \tag{4.4}$$

where the constant 2 cannot be improved.
This estimate allows one to deduce the uniqueness of the solution to (4.1) for a different (and often more interesting) class of functions $\mathbf{f}$ than before because we now allow $\mathbf{f}$ to be much larger near the boundary.

If our function $\mathbf{f}$ is large near a particular point $a \in \Omega$ we may use the well known Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{|x-a|^2} dx, \quad u \in \overset{\circ}{W}{}^{1,2}(\Omega), \quad \Omega \subset \mathbb{R}^3$$

to get a similar estimate

$$|||\mathbf{f}||| \leq 2|||x-a|\mathbf{f}||_{L^2}$$

useful for such functions.
Remark : Note that the L^2 vector versions of the Hardy inequalities used above, may be proved simply by adding the corresponding inequalities for scalar valued functions. This in particular implies that we may use different estimates for the various components of the vector $\mathbf{f}$.
If for example $\mathbf{f}=(f_1,f_2,f_3)$, where f_1 is large near the boundary, f_2 is large near a point $a\in\Omega$ and f_3 is large near a point $b\in\Omega$ this type of estimate is precisely what we need :

$$|||\mathbf{f}||| \leq 2(||\delta f_1||_{L^2} + |||x-a|f_2||_{L^2} + |||x-b|f_3||_{L^2})$$

One can also deduce the existence of a solution to (4.1) by using Hardy inequalities. This example is taken from [16] :
Let Ω be an arbitrary domain (unbounded domains are also allowed) and suppose that

$$\int_\Omega |x-y|^2 \sum_{k=1}^{3} |f_k(x)|^2 dx < \infty \text{ for some } y,$$

then

$$\left|\int_\Omega \mathbf{f}\cdot\mathbf{w}dx\right| \leq \left(\int_\Omega |x-y|^2\sum_{k=1}^{3}|f_k(x)|^2dx\right)^{\frac{1}{2}}\cdot\left(\int_\Omega\sum_{k=1}^{3}\frac{|w_i|^2}{|x-y|^2}dx\right)^{\frac{1}{2}} \leq C||\mathbf{w}||_{\mathbf{H}}$$

Hence, the condition of theorem 4.1 is satisfied so there exists at least one generalized solution to (4.1). One can modify this example in the same way as we did in our previous calculations :
Let Ω be a domain with Hardy constant $H(\Omega)>0$ and suppose that

$$\int_\Omega \delta(x)^2 \sum_{k=1}^{3} |f_k(x)|^2 dx = C < \infty$$

then

$$\left|\int_\Omega \mathbf{f}\cdot\mathbf{w}dx\right| \leq C^{\frac{1}{2}}\left(\int_\Omega\sum_{k=1}^{3}\frac{|w_i(x)|^2}{\delta(x)^2dx}\right)^{\frac{1}{2}} \leq C^{\frac{1}{2}}||\mathbf{w}||_{\mathbf{H}(\Omega)}$$

so we have a unique solution also in this case.
If different components of $\mathbf{f}$ are large at different places of Ω (i.e one component might be large on the boundary and another large at some point a and so on) we we may combine these two type of estimates to deduce the existence of solution also in this case. Also, functions $\mathbf{f}$ where the components have (sufficently weak) singularities at a finite number of points and are large at $\partial\Omega$ can be allowed, since we can write such a component as a sum of functions $\{g_i\}, i=1,\ldots,m$ where each g_i is large near just one point x_i (or near $\partial\Omega$). Then we use the triangle ineq. and estimate each g_i separately.

4.2 refined estimates

If we apply our refined Hardy inequality (1.10) we are able to improve the results concerning uniqueness in the last section when the volume of Ω is small.

Theorem 4.3. *The solution of 4.1 is unique if*

$$\frac{16\sqrt{3}\lambda_1^{-\frac{1}{4}}\nu^{-2}}{\left(\frac{(48\pi^2)^{\frac{1}{6}}}{|\Omega|^{\frac{1}{3}}||\mathbf{f}||_{L^2}}+\frac{1}{||\delta\mathbf{f}||_{L^2}}\right)^2}<1 \tag{4.5}$$

Proof. Let $0 \le b \le 1$. We get

$$\left|\int_\Omega \mathbf{f}\cdot\mathbf{w}dx\right| \le \left(b\int_\Omega |\delta\mathbf{f}|\frac{|\mathbf{w}|}{\delta}dx+(1-b)\int_\Omega|\mathbf{f}||\mathbf{w}|dx\right)^2$$

$$\le\left(b\left(\int_\Omega|\delta\mathbf{f}|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega\frac{|\mathbf{w}|^2}{\delta^2}dx\right)^{\frac{1}{2}}+(1-b)\left(\int_\Omega|\mathbf{f}|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega|\mathbf{w}|^2dx\right)^{\frac{1}{2}}\right)^2$$

$$\le 2b^2||\delta\mathbf{f}||_{L^2}^2||\frac{\mathbf{w}}{\delta}||_{L^2}^2+2(1-b)^2||\mathbf{f}||_{L^2}^2||\mathbf{w}||_{L^2}^2$$

Now put

$$b=\frac{||\mathbf{f}||_{L^2}}{||\mathbf{f}||_{L^2}+\frac{||\delta\mathbf{f}||_{L^2}(48\pi^2)^{\frac{1}{6}}}{|\Omega|^{\frac{1}{3}}}}.$$

This gives

$$\left|\int_\Omega\mathbf{f}\cdot\mathbf{w}dx\right|\le 8\left(\frac{(48\pi^2)^{\frac{1}{6}}}{|\Omega|^{\frac{1}{3}}||\mathbf{f}||_{L^2}}+\frac{1}{||\delta\mathbf{f}||_{L^2}}\right)^{-2}\left(\frac{1}{4}||\frac{\mathbf{w}}{\delta}||_{L^2}^2+\frac{3^{\frac{1}{3}}(4\pi)^{\frac{2}{3}}}{4|\Omega|^{\frac{2}{3}}}||\mathbf{w}||_{L^2}^2\right).$$

Using the Hardy ineq. 1.10 (for $n=3, p=2$), we obtain

$$|||\mathbf{f}|||\le 8\left(\frac{(48\pi^2)^{\frac{1}{6}}}{|\Omega|^{\frac{1}{3}}||\mathbf{f}||_{L^2}}+\frac{1}{||\delta\mathbf{f}||_{L^2}}\right)^{-2}.$$

Now just combine this result with theorem 4.2. ☐

λ_1 here of course depends on Ω, but one can obtain various lower bounds depending on the volume of Ω. λ_1 grows when $|\Omega|$ decreases.
Note that (4.5) is an improvement of the inequality (4.4) occurring in [21] when the volume of Ω is small.

If we analogously apply the Hardy ineq. (see for example [4])

$$\int_\Omega|\nabla u|dx\ge\frac{1}{4}\int_\Omega\frac{|u|^2}{|x-a|^2}+\Lambda_2\left(\frac{4\pi}{3|\Omega|}\right)^{\frac{2}{3}}\int_\Omega|u|^2dx,\quad u\in\overset{\circ}{W}{}^{1,2}(\Omega)$$

instead (where Λ_1 is the smallest eigenvalue for the Dirichlet Laplacian in the 2D unit disc) we get uniqueness of the solution also if

$$\frac{16\sqrt{3}\lambda_1^{-\frac{1}{4}}\nu^{-2}}{\left(\frac{(48\pi^2)^{\frac{1}{6}}}{|\Omega|^{\frac{1}{3}}||\mathbf{f}||_{L^2}}+\frac{1}{|||x-a|\mathbf{f}||_{L^2}}\right)^2}<1.$$

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