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A SEMILINEAR SCHRÖDINGER EQUATION IN THE PRESENCE OF A MAGNETIC FIELD

GIANNI ARIOLI AND ANDRZEJ SZULKIN

Abstract. We consider the semilinear stationary Schrödinger equation in a magnetic field $(-i\nabla + A)^2 u + V(x)u = g(x, |u|)u$ in $\mathbb{R}^N$, where V is the scalar (or electric) potential and A is the vector (or magnetic) potential. We study the existence of nontrivial solutions both in the critical and in the subcritical case (respectively $g(x, |u|) = |u|^{2^*-2}$ and $|g(x, |u|)| \le c(1 + |u|^{p-2})$, where $2 < p < 2^*$). The results are obtained by variational methods. For g critical we use constrained minimization and for subcritical g we employ a minimax-type argument. In the latter case we also study the existence of infinitely many geometrically distinct solutions.

1. Introduction

In this paper we study the existence of solutions $u \neq 0$ of the semilinear Schrödinger equation

$$(-i\nabla + A)^2 u + V(x)u = g(x, |u|)u, \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $u : \mathbb{R}^N \to \mathbb{C}$ and $N \ge 2$. Here $V : \mathbb{R}^N \to \mathbb{R}$ is the scalar (or electric) potential and $A = (A_1, \dots, A_N) : \mathbb{R}^N \to \mathbb{R}^N$ is the vector (or magnetic) potential. Let $B := \operatorname{curl} A$. For $N = 3$ this is the usual curl operator and for general N, $B = (B_{jk})$, $1 \le j, k \le N$, where $B_{jk} := \partial_j A_k - \partial_k A_j$. One can also consider A as a 1-form:

$$A = \sum_{j=1}^N A_j dx^j;$$

then $B = dA$, i.e.

$$B = \sum_{j<k} B_{jk} dx^j \wedge dx^k,$$

where B_{jk} are as above. B represents the external magnetic field whose source is A. Suppose $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$, $V \in L^1_{loc}(\mathbb{R}^N)$ and let V be bounded below. Denote

$$\nabla_A u = (\nabla + iA)\, u,$$

let

$$H^1_A(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N)\right\}$$

and, for $N \ge 3$,

$$\mathcal{D}^{1,2}_A(\mathbb{R}^N) := \left\{u \in L^{2^*}(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N)\right\},$$

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where $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent. Both $H^1_A(\mathbb{R}^N)$ and $\mathcal{D}^{1,2}_A(\mathbb{R}^N)$ are Hilbert spaces with inner product respectively

$$\int_{\mathbb{R}^N} \nabla_A u \cdot \overline{\nabla_A v} + u\overline{v}$$

and

$$\int_{\mathbb{R}^N} \nabla_A u \cdot \overline{\nabla_A v}$$

(the bar denotes complex conjugation). By Section 2 of [EL] and Theorem 7.22 of [LL], $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1_A(\mathbb{R}^N)$ and $\mathcal{D}^{1,2}_A(\mathbb{R}^N)$ (in [EL] $\mathcal{D}^{1,2}_A(\mathbb{R}^N)$ has in fact been defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm corresponding to the inner product above).

Let $|g(x,|u|)| \le c(1+|u|^{2^*-2})$ and $F(x,|u|) := \int_0^{|u|} g(x,s)s\,ds$; consider the functional

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2 - \int_{\mathbb{R}^N} F(x,|u|).$$

Suppose $u \in H^1_A(\mathbb{R}^N)$. By the diamagnetic inequality [LL, Theorem 7.21] (see also the next section), $|u| \in H^1(\mathbb{R}^N)$ and therefore $u \in L^p(\mathbb{R}^N)$ for any $p \in [2, 2^*]$ (for any $p \in [2,+\infty)$ if $N=2$). It follows that whenever $V \in L^\beta(\mathbb{R}^N)$, where $\beta \ge N/2$ ($\beta > 1$ if $N=2$), then $J \in C^1(H^1_A(\mathbb{R}^N), \mathbb{R})$ and critical points of J are weak solutions of (1.1). We note for further reference that $J(e^{i\vartheta}u) = J(u)$ for any $\vartheta \in \mathbb{R}$, hence J is S^1-invariant.

Suppose now $A, \tilde{A} \in L^\alpha_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ for some $\alpha \in [1,+\infty)$ and $\operatorname{curl} A = B = \operatorname{curl} \tilde{A}$ (in the sense of distributions). Then $\tilde{A} - A = \nabla\varphi$ for some $\varphi \in W^{1,\alpha}_{loc}(\mathbb{R}^N)$, see [L, Lemma 1.1]. It is easy to see that if $\tilde{u} = e^{-i\varphi}u$, then $\nabla_{\tilde{A}}\tilde{u} = e^{-i\varphi}\nabla_A u$ and hence

$$\int_{\mathbb{R}^N} |\nabla_{\tilde{A}}\tilde{u}|^2 = \int_{\mathbb{R}^N} |\nabla_A u|^2 .$$

It follows that if $u \in H^1_A(\mathbb{R}^N)$, then $\tilde{u} \in H^1_{\tilde{A}}(\mathbb{R}^N)$ and if u satisfies (1.1), then so does $\tilde{u}$ with A replaced by $\tilde{A}$. The above properties are called the gauge invariance and the transformation $u \mapsto \tilde{u}$ the change of gauge. These properties are consistent with the fact that the magnetic field B and not the particular choice of the vector potential A should be essential. Note that there is a trivial change of gauge $u \mapsto \tilde{u} = e^{-i\vartheta}u$, where ϑ is a constant. Then $\tilde{A} = A$ and in fact this property gives rise to the S^1-invariance of J mentioned above. While there is a vast literature concerning the Schrödinger equation (1.1) with $A = 0$, to the best of our knowledge there are only a few papers dealing with the magnetic case [EL, P, ST]. Also in [CS, K] the magnetic case has been considered, but from a very different point of view (semiclassical limits and related concentration phenomena).

Denote $-\Delta_A := (-i\nabla + A)^2$. Since V is bounded below, so is the spectrum $\sigma(-\Delta_A + V)$ in $L^2(\mathbb{R}^N)$. Suppose $F \ge 0$. If $0 \notin \sigma(-\Delta_A + V)$, then either $\sigma(-\Delta_A + V) \subset (0,+\infty)$ and the functional has a mountain pass geometry, or $\sigma(-\Delta_A + V) \cap (-\infty, 0) \neq \emptyset$ and then J has a geometry of linking type.

Now we proceed to formulate our main results. First we consider a minimization problem in $\mathbb{R}^N$, $N \geq 3$. Let

$$\bar{S} := \inf_{u \in \mathcal{D}^{1,2}_A(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2}{||u||^2_{2^*}}, \tag{1.2}$$

where here and in what follows $||\cdot||_p$ denotes the usual L^p-norm in $\mathbb{R}^N$.

Theorem 1.1. *If $V \geq 0$, $V \in L^{N/2}_{loc}(\mathbb{R}^N)$ and $A \in L^N_{loc}(\mathbb{R}^N, \mathbb{R}^N)$, then the infimum in (1.2) is attained if and only if $V \equiv 0$ and $B = \operatorname{curl} A \equiv 0$.*

The above result is a slight generalization of Theorem 3.7 in [EL], but our proof is considerably simpler. Note that since $V \in L^{N/2}_{loc}(\mathbb{R}^N)$ only, then $\int_{\mathbb{R}^N} V|u|^2$ need not be finite for all $u \in \mathcal{D}^{1,2}_A(\mathbb{R}^N)$. However, it is finite for $u \in C^\infty_0(\mathbb{R}^N)$ and therefore the minimization problem (1.2) makes sense. Note also that if $\bar{S}$ is attained at some $u \in \mathcal{D}^{1,2}_A(\mathbb{R}^N)$, then u is a solution of (1.1) with $g(x, |u|) = \bar{S}|u|^{2^*-2}$; hence $v = \bar{S}^{1/(2^*-2)}u$ solves (1.1) with $g(x, |u|) = |u|^{2^*-2}$.

Theorem 1.2. *Suppose $N \geq 4$, $V \in L^1_{loc}(\mathbb{R}^N)$, $V^- := \max\{-V, 0\} \in L^{N/2}(\mathbb{R}^N)$, $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ and $\sigma(-\Delta_A + V) \subset (0, +\infty)$, where $\sigma(\cdot)$ denotes the spectrum in $L^2(\mathbb{R}^N)$. If there exists $\bar{x} \in \mathbb{R}^N$ such that $V(x) \leq -c < 0$ in a neighborhood of $\bar{x}$ and A is continuous at $\bar{x}$, then the infimum of (1.2) is attained for some $u \in H^1_A(\mathbb{R}^N) \setminus \{0\}$.*

Since $\sigma(-\Delta_A + V) \subset (0, +\infty)$, it follows that if $\bar{S}$ is attained, then it is positive. Indeed, $\bar{S} \geq 0$ and if $\bar{S} = 0$ is attained at some $u \neq 0$, then u is an eigenvalue of $-\Delta_A + V$ which is impossible.

For the next theorem we introduce the following assumptions:

A1: $V \in L^\infty(\mathbb{R}^N)$, $g \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$ and $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$.
A2: V, g and $B = \operatorname{curl} A$ are 1−periodic in x_j, $1 \leq j \leq N$.
A3: $g(x, 0) = 0$.
A4: There are constants $C > 0$ and $p \in (2, 2^*)$ ($p > 2$ if $N = 2$) such that $|g(x, |u|)| \leq C(1 + |u|^{p-2})$ for all x, u.
A5: There is a constant $\mu > 2$ such that $0 < \mu F(x, |u|) \leq g(x, |u|)|u|^2$ whenever $u \neq 0$.

Note that in view of the definition of F, (A5) is the usual superlinearity condition. Since $B_{jk} = \partial_j A_k - \partial_k A_j$ in the sense of distributions, the periodicity of B should be interpreted as $B(\cdot) - B(\cdot + e_j)$ being the zero distribution for any element e_j of the standard basis in $\mathbb{R}^N$. It is also clear that according to (A3), (1.1) has the trivial solution $u = 0$.

Theorem 1.3. *If $0 \notin \sigma(-\Delta_A + V)$ and conditions (A1)-(A5) are satisfied, then equation (1.1) has a nontrivial solution $u \in H^1_A(\mathbb{R}^N)$.*

A corresponding result is well-known for the Schrödinger equation with $A = 0$ (see e.g. [KS, W] and the references there).

Finally we shall exploit the S^1−invariance of J in order to show the existence of infinitely many solutions of (1.1). For this purpose we introduce one more assumption:

A6: There are constants $\bar{C}, \varepsilon_0 > 0$ such that

$$|g(x, |u + v|)(u + v) - g(x, |u|)u| \leq \bar{C}|v|(1 + |u|^{p-1})$$

whenever $|v| \leq \varepsilon_0$, where p is as in (A4).

Theorem 1.4. *If $0 \notin \sigma(-\Delta_A + V)$ and conditions (A1)-(A6) are satisfied, then equation (1.1) has infinitely many geometrically distinct solutions.*

By geometrically distinct we mean such u, v that $v \neq e^{i\vartheta}u$ for any $\vartheta \in \mathbb{R}$ and $T_z v \neq u$ for any $z \in \mathbb{Z}^N$, where T_z is a certain operator corresponding to the translation by elements of $\mathbb{Z}^N$ in the nonmagnetic case. A more precise definition will be given in Section 4.

The above result should be compared to the one contained in [BD, KS], where A was equal to 0.

The paper is organized as follows. In Section 2 we prove some auxiliary results, Section 3 deals with the minimization problem (1.2) and Theorems 1.1 and 1.2, the proof of Theorem 1.3 is given in Section 4, and in Section 5 we prove Theorem 1.4. In the appendix we sketch a proof of Lemma 3.1, which is the magnetic version of a concentration-compactness result.

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2. Preliminary lemmas

The following well-known *diamagnetic inequality* is proved in [LL]:

Theorem 2.1. *If $u \in H^1_A(\mathbb{R}^N)$ (resp. $u \in \mathcal{D}^{1,2}_A(\mathbb{R}^N)$), then $|u| \in H^1(\mathbb{R}^N)$ (resp. $|u| \in \mathcal{D}^{1,2}(\mathbb{R}^N)$) and*

$$\left|\nabla |u|(x)\right| \leq |\nabla u(x) + iA(x)u(x)| \ \text{ for a.e. } x \in \mathbb{R}^N.$$

Proof. We outline the argument. See [LL] for more details. Since A is real-valued,

$$\left|\nabla |u|(x)\right| = \left|\operatorname{Re}\left(\nabla u \frac{\bar{u}}{|u|}\right)\right| = \left|\operatorname{Re}\left((\nabla u + iAu)\frac{\bar{u}}{|u|}\right)\right| \leq |\nabla u + iAu|. \tag{2.1}$$

□

Remark 2.2. The spaces $H^1_A(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ are incomparable; more precisely, in general $H^1_A(\mathbb{R}^N) \not\subseteq H^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \not\subseteq H^1_A(\mathbb{R}^N)$. On the other hand, when restricted to a bounded set they are equivalent, as stated in the following lemma (which will not be used later).

Lemma 2.3. *Suppose $A \in L^\alpha_{loc}(\mathbb{R}^N, \mathbb{R}^N)$, where $\alpha = N$ if $N \geq 3$ and $\alpha > 2$ if $N = 2$. If Ω is an open bounded subset of $\mathbb{R}^N$ with regular boundary, then $u \in H^1_A(\Omega)$ if and only if $u \in H^1(\Omega)$. Moreover, there exist $c_1, c_2 > 0$ only depending on Ω such that $c_1 \|u\|_{H^1(\Omega)} \leq \|u\|_{H^1_A(\Omega)} \leq c_2 \|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$.*

A regular boundary means here that the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous (a sufficient condition for regularity is that $\partial\Omega$ is Lipschitz continuous).

Proof. We consider the case $N \geq 3$ and leave the other one to the reader. By the Hölder inequality and the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ we have

$$\int_\Omega |Au|^2 \leq \|A\|^2_{L^N(\Omega)} \|u\|^2_{L^{2^*}(\Omega)} \leq c \|A\|^2_{L^N(\Omega)} \|u\|^2_{H^1(\Omega)},$$

hence

$$\int_\Omega |\nabla_A u|^2 + |u|^2 \le \int_\Omega 2(|\nabla u|^2 + |Au|^2) + |u|^2 \le \tilde{c} \int_\Omega |\nabla u|^2 + |u|^2.$$

To prove the other inequality (i.e. $c_1 \, ||u||_{H^1(\Omega)} \le ||u||_{H^1_A(\Omega)}$), note that

$$\begin{aligned}\int_\Omega |\nabla_A u|^2 + |u|^2 &\ge \int_\Omega \big||\nabla u| - |Au|\big|^2 + |u|^2 \\ &= \int_\Omega |\nabla u|^2 - 2|Au||\nabla u| + |A|^2|u|^2 + |u|^2.\end{aligned}$$

It remains to show that

$$\int_\Omega |\nabla u|^2 - 2|Au||\nabla u| + |A|^2|u|^2 + |u|^2 \ge \varepsilon \int_\Omega |\nabla u|^2 + |u|^2$$

for some positive ε. Arguing indirectly, we find u_n with $\|u_n\|_{H^1(\Omega)} = 1$ such that

$$\int_\Omega |\nabla u_n|^2 - 2|Au_n||\nabla u_n| + |A|^2|u_n|^2 + |u_n|^2 < \frac{1}{n}.$$

Passing to a subsequence, $u_n \rightharpoonup u$ in $H^1(\Omega)$ and since $A \in L^N(\Omega)$, $\int_\Omega |Au_n| \, |\nabla u_n| \to \int_\Omega |Au| \, |\nabla u|$. Hence

$$\int_\Omega \big||\nabla u| - |Au|\big|^2 + |u|^2 = \int_\Omega |\nabla u|^2 - 2|Au||\nabla u| + |A|^2|u|^2 + |u|^2 \le 0.$$

If $u \ne 0$, this is a contradiction, and if $u = 0$, then the inequality above is strict because $u_n \not\to 0$ in $H^1(\Omega)$, a contradiction again. □

The following result is a special case of Proposition 2.2 in [EL] (see also Remark 2.4 there):

Proposition 2.4. *If $A \in W^{1,\infty}_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ and either $\partial_j A_k - \partial_k A_j \ge c$ a.e. in $\mathbb{R}^N$ or $\partial_j A_k - \partial_k A_j \le -c$ a.e. in $\mathbb{R}^N$ for some $j, k \in \{1, \dots, N\}$ and $c > 0$, then*

$$c\|u\|_2^2 \le \|(\partial_j + iA_j)u\|_2^2 + \|(\partial_k + iA_k)u\|_2^2 \le \|\nabla_A u\|_2^2$$

for all $u \in H^1_A(\mathbb{R}^N)$.

In Theorem 1.2 we require that $\sigma(-\Delta_A + V) \subset (0, +\infty)$. According to Lemma 2.5 below, either of the following assumptions suffices.

P1: $A \in L^N_{loc}(\mathbb{R}^N)$, $V \in L^1_{loc}(\mathbb{R}^N)$, where $N \ge 3$, and there exists a bounded set $\Omega \subset \mathbb{R}^N$ and two constants $c_1, c_2 > 0$ such that

(2.2) $$V(x) \ge c_1 \text{ for all } x \notin \Omega$$

and

(2.3) $$\inf_{x \in \Omega} V(x) = -c_2 > -S\mu(\Omega)^{-2/N},$$

S being the Sobolev constant for the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

P2: $A \in W^{1,\infty}_{loc}(\mathbb{R}^N, \mathbb{R}^N)$, $V \in L^1_{loc}(\mathbb{R}^N)$ and there exists a constant $c_3 > 0$ such that $\inf_{x \in \mathbb{R}^N} V(x) > -c_3$ and either $\partial_j A_k - \partial_k A_j \ge c_3$ a.e. in $\mathbb{R}^N$ or $\partial_j A_k - \partial_k A_j \le -c_3$ a.e. in $\mathbb{R}^N$ for some $j, k \in \{1, \dots, N\}$.

Lemma 2.5. *If P1 or P2 above holds, then there exists $\varepsilon > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2 \geq \varepsilon \|u\|^2_{H^1_A(\mathbb{R}^N)}$$

for all $u \in H^1_A(\mathbb{R}^N)$ and therefore $\sigma(-\Delta_A + V) \subset (0, +\infty)$.

Proof. P1. Note that if $V|u|^2 \notin L^1(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} V|u|^2 = +\infty$, so the first conclusion is trivially satisfied. We may therefore assume $V|u|^2 \in L^1(\mathbb{R}^N)$.

By the Hölder inequality,

$$\int_\Omega |u|^2 \leq \mu(\Omega)^{2/N} \|u\|^2_{2^*},$$

and by the Sobolev and the diamagnetic inequalities,

$$S\|u\|^2_{2^*} \leq \int_{\mathbb{R}^N} \big|\nabla |u|\big|^2 \leq \int_{\mathbb{R}^N} |\nabla_A u|^2,$$

therefore

$$\int_\Omega |u|^2 \leq S^{-1}\mu(\Omega)^{2/N} \int_{\mathbb{R}^N} |\nabla_A u|^2. \tag{2.4}$$

By (2.3) and (2.4) we have

$$\int_\Omega V|u|^2 \geq -c_2 \int_\Omega |u|^2 \geq -c_2 S^{-1}\mu(\Omega)^{2/N} \int_{\mathbb{R}^N} |\nabla_A u|^2. \tag{2.5}$$

Suppose now that the first conclusion does not hold. Then we can find a sequence $\{u_n\}$ such that $\|u_n\|_{H^1_A(\mathbb{R}^N)} = 1$ and $\int_{\mathbb{R}^N} |\nabla_A u_n|^2 + V|u_n|^2 \to 0$. Let $\tilde{c} := 1 - c_2 S^{-1}\mu(\Omega)^{2/N}$, then $\tilde{c} > 0$ according to (2.3), and by (2.2) and (2.5),

$$\tilde{c} \int_{\mathbb{R}^N} |\nabla_A u_n|^2 + c_1 \int_{\Omega^c} |u_n|^2 \leq \int_{\mathbb{R}^N} |\nabla_A u_n|^2 + \int_\Omega V|u_n|^2 + \int_{\Omega^c} V|u_n|^2 \to 0.$$

Hence $\|u_n\|_{L^2(\Omega^c)} \to 0$ and $\|\nabla_A u_n\|_2 \to 0$. Since it follows from (2.4) that also $\|u_n\|_{L^2(\Omega)} \to 0$, the sequence $\{u_n\}$ tends to 0 in $H^1_A(\mathbb{R}^N)$ which is a contradiction.

P2. Choose $\varepsilon > 0$ such that $\inf_{x\in\mathbb{R}^N} V(x) \geq -c_3 + \varepsilon(1 + c_3)$. By Proposition 2.4, $\|\nabla_A u\|^2_2 \geq c_3\|u\|^2_2$, hence

$$\int_{\mathbb{R}^N} (1-\varepsilon)|\nabla_A u|^2 + (V - \varepsilon)|u|^2 \geq 0$$

and the conclusion follows. □

It is clear that the assumptions P1 and P2 are not necessary for $\sigma(-\Delta_A + V)$ to be contained in $(0, +\infty)$. However, they illustrate how A and V can be chosen in order to satisfy the hypotheses of Theorem 1.2. Note also that the first conclusion of the lemma shows the quadratic form associated with $-\Delta_A + V$ is positive definite in $H^1_A(\mathbb{R}^N)$ which is more than we need.

We shall make repeated use of the following fact:

Lemma 2.6. *Let $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ and suppose $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}_A(\mathbb{R}^N)$. Then, up to a subsequence, $u_n \to u$ a.e. in $\mathbb{R}^N$ and $u_n \to u$ in $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [2, 2^*)$. The same conclusion holds if $u_n \rightharpoonup u$ in $H^1_A(\mathbb{R}^N)$.*

Proof. By the diamagnetic inequality the injection $\mathcal{D}^{1,2}_A(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous, hence $u_n \rightharpoonup u$ in $L^{2^*}(\mathbb{R}^N)$. Moreover, $|u_n - u|$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. So passing to a subsequence, $u_n \to u$ a.e. and $|u_n - u| \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. It follows from the Rellich-Kondrachov theorem that $u_n \to u$ in $L^q_{loc}(\mathbb{R}^N)$. The second part of the lemma is proved similarly. □

3. The critical case

Proof of Theorem 1.1. Necessary condition. We first prove that $\bar{S}$ is the Sobolev constant. Indeed, by the Sobolev and the diamagnetic inequalities we have

$$S \le \frac{\int_{\mathbb{R}^N} \left|\nabla |u|\right|^2 + V|u|^2}{||u||^2_{2^*}} \le \frac{\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2}{||u||^2_{2^*}},$$

therefore $S \le \bar{S}$. Let

$$U_\varepsilon(x) = (N(N-2))^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N-2}{2}} \tag{3.1}$$

and $u_\varepsilon(x) = \psi(x)U_\varepsilon(x)$, where $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$, $\psi = 1$ on $B(0, 1/2)$ and $\psi = 0$ off $B(0,1)$ ($B(a,r)$ is the open ball of radius r centered at a). Then

$$\|\nabla(\psi U_\varepsilon)\|_2^2 \equiv \|\nabla u_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}) \text{ and } \|u_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N) \tag{3.2}$$

(see e.g. [W], p. 35). Since u_ε is bounded in $L^{2^*}(\mathbb{R}^N)$ and $u_\varepsilon \to 0$ a.e., $u_\varepsilon \rightharpoonup 0$ in $L^{2^*}(\mathbb{R}^N)$ as $\varepsilon \to 0$. Therefore $\int_{\mathbb{R}^N} V|u_\varepsilon|^2 = \langle V, |u_\varepsilon|^2\rangle \to 0$ as $\varepsilon \to 0$, where the duality product is taken with respect to $L^{N/2}(\mathbb{R}^N)$ and $L^{2^*/2}(\mathbb{R}^N)$. By the same argument,

$$\int_{\mathbb{R}^N} |Au_\varepsilon|^2 = \langle |A|^2, |u_\varepsilon|^2\rangle \to 0 \text{ as } \varepsilon \to 0.$$

Let $\delta > 0$. Choosing ε small enough we have

$$\frac{\int_{\mathbb{R}^N} |\nabla_A u_\varepsilon|^2 + V|u_\varepsilon|^2}{\|u_\varepsilon\|^2_{2^*}} = \frac{\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + |Au_\varepsilon|^2 + V|u_\varepsilon|^2}{\|u_\varepsilon\|^2_{2^*}} \le S + \delta$$

(recall that u_ε is a real function), therefore $\bar{S} \le S$. Now assume that u is a minimizer normalized by $\|u\|_{2^*} = 1$. Then

$$S = \int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2 \ge \int_{\mathbb{R}^N} |\nabla_A u|^2 \ge \int_{\mathbb{R}^N} \left|\nabla |u|\right|^2 \ge S$$

and it follows that $|u(x)| = U_\varepsilon(x-a)/\|U_\varepsilon\|_{2^*}$ for some $a \in \mathbb{R}^N$ (that the minimizer for $\|\nabla u\|_2^2/\|u\|_{2^*}^2$ is unique up to translation and dilation can be seen e.g. from the proof of Theorem 1.42 in [W]). In particular, $|u| > 0$ for all x and therefore $V \equiv 0$. Moreover, the inequality of Theorem 2.1 must be an equality a.e. So by (2.1), the imaginary part of $(\nabla u + iAu)\bar{u}$ must be zero which is equivalent to $A = -\mathrm{Im}\,(\nabla u/u)$. An easy computation shows that $\mathrm{curl}\,(\nabla u/u) = 0$.

Sufficient condition. Assume that $V \equiv 0$ and $\mathrm{curl}\, A = 0$. Then $A = \nabla\varphi$ for some $\varphi \in W^{1,N}_{loc}(\mathbb{R}^N)$ according to [L] and it is easy to verify that $u = U_\varepsilon e^{-i\varphi}$ is a minimizer for (1.2) for any $\varepsilon > 0$. □

In order to study the compactness of minimizing sequences we adapt the concentration-compactness technique of [W] (see Lemma 1.40 there) as follows:

Lemma 3.1. *Suppose $N \geq 3$ and $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$. Let $\{u_n\} \subset \mathcal{D}^{1,2}_A(\mathbb{R}^N)$ be a sequence such that*

(i) $u_n \rightharpoonup u$ *in* $\mathcal{D}^{1,2}_A(\mathbb{R}^N)$
(ii) $|\nabla_A(u_n - u)|^2 \rightharpoonup \mu$ *in* $\mathcal{M}(\mathbb{R}^N)$ *($\mathcal{M}(\mathbb{R}^N)$ denotes the space of finite measures)*
(iii) $|u_n - u|^{2^*} \rightharpoonup \nu$ *in* $\mathcal{M}(\mathbb{R}^N)$
(iv) $u_n \to u$ *a.e. in* $\mathbb{R}^N$.

Define $\mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla_A u_n|^2$ *and* $\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n|^{2^*}$. *Then*

(1) $||\nu||^{2/2^*} \leq S^{-1} ||\mu||$
(2) $\nu_\infty^{2/2^*} \leq S^{-1}\mu_\infty$
(3) $\limsup ||\nabla_A u_n||_2^2 = ||\nabla_A u||_2^2 + ||\mu|| + \mu_\infty$
(4) $\limsup ||u_n||_{2^*}^{2^*} = ||u||_{2^*}^{2^*} + ||\nu|| + \nu_\infty$.

Moreover, if $u = 0$ and $||\nu||^{2/2^*} = S^{-1} ||\mu||$, then μ and ν are concentrated at a single point.

The proof follows closely [W] once the results in Section 2 are taken into account. A sketch of it is given in the Appendix.

Proof of Theorem 1.2. With no restriction assume that $\bar{x} = 0$. Let $\vartheta(x) := -\sum A_j(0)x_j$. Then $(A + \nabla\vartheta)(0) = 0$ and by continuity $|(A + \nabla\vartheta)(x)|^2 \leq c' < c$ for all $|x| < \delta$. Possibly choosing a smaller δ we have $V(x) \leq -c$ for all $|x| < \delta$. Let U_ε be as in (3.1) and let $v_\varepsilon(x) := \psi(x)U_\varepsilon(x)e^{i\vartheta(x)}$, where $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$, $\psi(x) = 1$ in $B(0, \delta/2)$ and $\psi(x) = 0$ when $|x| \geq \delta$. Using (3.2) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_A v_\varepsilon|^2 + V|v_\varepsilon|^2 &\leq \int_{\mathbb{R}^N} |\nabla(\psi U_\varepsilon)|^2 + \psi^2 U_\varepsilon^2 |\nabla\vartheta + A|^2 - c\psi^2 U_\varepsilon^2 \\ &\leq S^{\frac{N}{2}} + (c' - c)\int_{B(0,\delta/2)} U_\varepsilon^2 + O(\varepsilon^{N-2})\end{aligned}$$

and $\|v_\varepsilon\|_{2^*}^2 = S^{(N-2)/2} + O(\varepsilon^N)$. It is a standard result that for small $\varepsilon > 0$

$$\int_{B(0,\delta/2)} U_\varepsilon^2 \geq \begin{cases} C\varepsilon^2|\log\varepsilon| & \text{if } N = 4 \\ C\varepsilon^2 & \text{if } N \geq 5, \end{cases}$$

where $C > 0$ (cf. e.g. [W], p. 35), hence

$$\bar{S} = \inf_{u\in\mathcal{D}^1_A(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2}{||u||_{2^*}^2} < S. \tag{3.3}$$

Let $\{u_n\}$ be a minimizing sequence normalized by $||u_n||_{2^*} = 1$. Then, taking a subsequence if necessary, $u_n \rightharpoonup u$ in $L^{2^*}(\mathbb{R}^N)$. Since $V^- \in L^{N/2}(\mathbb{R}^N)$, we have $\lim_{n\to\infty} \int_{\mathbb{R}^N} V^-|u_n|^2 = \int_{\mathbb{R}^N} V^-|u|^2$, so by Fatou's lemma,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^2 \geq \int_{\mathbb{R}^N} V|u|^2 \tag{3.4}$$

after passing to a subsequence. Therefore $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}_A(\mathbb{R}^N)$ and we may assume passing to a subsequence once more that (i)–(iv) of Lemma 3.1 are satisfied (cf. Lemma 2.6 and [W], Lemma 1.39).

We complete the proof by showing that $\|u\|_{2^*} = 1$. We have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla_A u_n|^2 + V|u_n|^2 = \bar{S} = \bar{S} \lim_{n\to\infty} ||u_n||_{2^*}^2,$$

hence by (3) and (4) of Lemma 3.1,

$$||\nabla_A u||_2^2 + ||\mu|| + \mu_\infty + \lim_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^2 = \bar{S}(||u||_{2^*}^{2^*} + ||\nu|| + \nu_\infty)^{2/2^*}.$$

Therefore, using (3.4),

$$\begin{aligned} ||\nabla_A u||_2^2 + ||\mu|| + \mu_\infty + \int_{\mathbb{R}^N} V|u|^2 &\le \bar{S}(||u||_{2^*}^{2^*} + ||\nu|| + \nu_\infty)^{2/2^*} \\ &\le \bar{S}(||u||_{2^*}^2 + ||\nu||^{2/2^*} + \nu_\infty^{2/2^*}). \end{aligned}$$

This and (1), (2) of Lemma 3.1 imply

$$||\nabla_A u||_2^2 + ||\mu|| + \mu_\infty + \int_{\mathbb{R}^N} V|u|^2 \le \bar{S}||u||_{2^*}^2 + \frac{\bar{S}}{S}\,||\mu|| + \frac{\bar{S}}{S}\mu_\infty.$$

Moreover, it follows from (3.3) that

$$||\nabla_A u||_2^2 + \int_{\mathbb{R}^N} V|u|^2 \ge \bar{S}\|u\|_{2^*}^2,$$

hence $\|\mu\| + \mu_\infty \le \frac{\bar{S}}{S}(\|\mu\| + \mu_\infty)$, and since $\bar{S} < S$, $\|\mu\| = \mu_\infty = \|\nu\| = \nu_\infty = 0$. Consequently, $||u||_{2^*} = 1$ and u is a minimizer. Since $\sigma(-\Delta_A + V) \subset (0, +\infty)$,

$$\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2 \ge \varepsilon\|u\|_2^2$$

for some $\varepsilon > 0$, so $u \in H^1_A(\mathbb{R}^N)$. □

4. Proof of Theorem 1.3

Suppose A and B satisfy the asssumptions A1 and A2. Then, for all $z \in \mathbb{Z}^N$, $\operatorname{curl} A(x+z) - \operatorname{curl} A(x) = B(x+z) - B(x) = 0$; hence

$$A(x+z) - A(x) = \nabla\varphi_z(x) \tag{4.1}$$

for some $\varphi_z \in H^1_{loc}(\mathbb{R}^N)$. In general A is not periodic, therefore the operator ∇_A is not translation invariant. However, in view of (4.1) we define a different "translation" $T : H^1_A(\mathbb{R}^N) \times \mathbb{Z}^N \to H^1_A(\mathbb{R}^N)$ by setting $(T_z u)(x) := u(x+z)e^{i\varphi_z(x)}$. Note that in general $T_{z_1+z_2} \ne T_{z_2}T_{z_1}$, hence T is not a group action of $\mathbb{Z}^N$. That the operator T is well-defined is a consequence of the following

Lemma 4.1. *Let $u \in H^1_A(\mathbb{R}^N)$, $z \in \mathbb{Z}^N$ and $v := T_z u$. Then $v \in H^1_A(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} |\nabla_A v|^2 = \int_{\mathbb{R}^N} |\nabla_A u|^2$ and $||v||_{H^1_A(\mathbb{R}^N)} = ||u||_{H^1_A(\mathbb{R}^N)}$. In particular, for each $z \in \mathbb{Z}^N$ the operator T_z is an isometry.*

Proof. Using (4.1), we have

$$\begin{aligned} \nabla_A v(x) &= \nabla\left(u(x+z)e^{i\varphi_z(x)}\right) + iA(x)u(x+z)e^{i\varphi_z(x)} \\ &= \left(\nabla u(x+z) + iA(x+z)u(x+z)\right)e^{i\varphi_z(x)}, \end{aligned}$$

therefore $\int_{\mathbb{R}^N} |\nabla_A v|^2 = \int_{\mathbb{R}^N} |\nabla_A u|^2$. Furthermore, $\int_{\mathbb{R}^N} |v(x)|^2 = \int_{\mathbb{R}^N} |u(x+z)|^2 = \int_{\mathbb{R}^N} |u(x)|^2$; hence the conclusion. □

Let

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2 - \int_{\mathbb{R}^N} F(x, |u|). \tag{4.2}$$

Although T is not a group action, we shall still say that J is invariant and J' equivariant with respect to the action of $\mathbb{Z}^N$ if $J(T_z u) = J(u)$ and $J'(T_z u) = T_z J'(u)$ for all $z \in \mathbb{Z}^N$.

Lemma 4.2. *J is invariant and J' is equivariant with respect to the action of $\mathbb{Z}^N$.*

Proof. The invariance of J follows from Lemma 4.1 and the periodicity of V and F. By this invariance,

$$\langle J'(u), v\rangle = \langle J'(T_z u), T_z v\rangle = \langle T_z^{-1} J'(T_z u), v\rangle.$$

Hence $T_z^{-1} J'(T_z u) = J'(u)$ and $J'(T_z u) = T_z J'(u)$. □

Proof of Theorem 1.3. Let $E := H^1_A(\mathbb{R}^N)$. Since $0 \notin \sigma(-\Delta_A + V)$, we can decompose E into the direct sum of two subspaces E^+ and E^- invariant with respect to $-\Delta_A + V$ and such that the quadratic form $\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2$ is positive (resp. negative) definite on E^+ (resp. E^-) (cf. [St], Section 8). Here we have used the fact that V is bounded and therefore the graph norm of $|-\Delta_A + V + \lambda|^{1/2}$, where $\lambda \geq \inf_{\mathbb{R}^N} V$, is equivalent to the $H^1_A(\mathbb{R}^N)$-norm. Since the quadratic form above is T_z-invariant, then the subspaces $E^\pm$ are also T_z-invariant. We have $\dim E^+ = +\infty$ and $\dim E^- = 0$ if $\sigma(-\Delta_A + V) \subset (0, +\infty)$, $\dim E^- = +\infty$ otherwise (the dimension must be infinite because $T_z E^- \subset E^-$ for all $z \in \mathbb{Z}^N$). We may introduce a more convenient equivalent norm in E in such a way that

$$\int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2 = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm \in E^\pm$ (if $\sigma(-\Delta_A + V) \subset (0, +\infty)$, then $E^- = \{0\}$ and $u^- = 0$).

If $\sigma(-\Delta_A + V) \subset (0, +\infty)$, it is a standard procedure to check that the functional J has a mountain pass geometry and that it admits a bounded Palais-Smale sequence $\{u_n\}$ at some positive level c (a $(PS)_c$-sequence for short). If instead $\sigma(-\Delta_A + V) \cap (-\infty, 0) \neq \emptyset$, then the functional has an infinite dimensional linking geometry as described in [KS]. More precisely, Lemmas 1.3 and 1.4 of [KS] apply here and they show that

$$d := \inf\{J(u) : u \in \partial B(0, \rho) \cap E^+\} > 0 \tag{4.3}$$

if $\rho > 0$ is small enough and $J(u) \leq 0$ on ∂M, where $M := \{u = u^- + sz_0 : u^- \in E^-,\ s \geq 0,\ \|u\| \leq R\}$, z_0 is a fixed element of $E^+ \cap \partial B(0,1)$ and $R > \rho$ is sufficiently large. Hence by Theorem 3.4 of [KS], J admits a $(PS)_c$-sequence $\{u_n\}$ for some $c \geq d > 0$. Moreover, $\{u_n\}$ is bounded by the argument of Lemma 1.5 in [KS]. The invariance of the functional with respect to T_z makes the Palais-Smale condition fail, nonetheless we can apply the concentration-compactness technique. By Lemma 1.7 in [KS], either $u_n \to 0$ in E (up to a subsequence) which is impossible because $J(u_n) \to c > 0$, or there exists a sequence $\{z_n\}$ in $\mathbb{Z}^N$ and $r, \eta > 0$ such that $\int_{B(z_n, r)} |u_n(x)|^2 \geq \eta$. Let $v_n := T_{z_n} u_n$. By Lemma 4.2, $\{v_n\}$ is also a $(PS)_c$-sequence, so (up to a subsequence again) $v_n \rightharpoonup v$ in E and $v_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$ (by Lemma 2.6). Hence $J'(v_n) \rightharpoonup J'(v)$ and v is a critical point of J; moreover, $v \neq 0$ because

$$\int_{B(0,r)} |v_n(x)|^2 = \int_{B(z_n, r)} |u_n(x)|^2 \geq \eta.$$

□

5. Proof of Theorem 1.4

Let J be as in (4.2) and let E, E^+ and E^- be as in the proof of Theorem 1.3. It is clear that J is invariant with respect to the representation S of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ given by $S_\vartheta u = e^{i\vartheta}u$ and we have already seen that J is invariant with respect to the (non-group) representation T of $\mathbb{Z}^N$. Note that $S_\vartheta T_z = T_z S_\vartheta$ and $E^{S^1} := \{u \in E : S_\vartheta u = u\} = \{0\}$ (in fact S^1 acts freely on $E \setminus \{0\}$). Let $\mathcal{O}_{S^1\times\mathbb{Z}^N}(u) := \{S_\vartheta T_z u : \vartheta \in S^1, z \in \mathbb{Z}^N\}$. Two solutions u, v of (1.1) are called *geometrically distinct* if they belong to different orbits, i.e. if $\mathcal{O}_{S^1\times\mathbb{Z}^N}(u) \neq \mathcal{O}_{S^1\times\mathbb{Z}^N}(v)$.

Our argument is a straightforward adaptation of that in [AS], therefore we only summarize the main steps and concentrate on pointing out the differences.

Suppose that (1.1) has only finitely many geometrically distinct solutions. Denote the set of critical points of J by K_J. Since $J(u) > 0$ if $u \in K_J \setminus \{0\}$ (cf. e.g. (4.1) in [KS]) and $J|_{E^-} \le 0$, $K_J \cap E^- = \{0\}$. Let $\mathcal{C}$ be a set consisting of arbitrarily chosen representatives of the orbits $\mathcal{O}_{S^1\times\mathbb{Z}^N}(u)$, $u \in K_J \setminus \{0\}$ and let $\mathcal{K} := \mathcal{O}_{S^1}(\mathcal{C}) = \{S_\vartheta u : \vartheta \in S^1,\ u \in \mathcal{C}\}$. Clearly, $\mathcal{K}$ is a compact set,

$$K_J \setminus \{0\} = \mathcal{O}_{\mathbb{Z}^N}(\mathcal{K}) \tag{5.1}$$

and if $\mathcal{F} := P_{E^+}(\mathcal{K})$, then

$$T_{z_1}\mathcal{F} \cap T_{z_2}\mathcal{F} = \emptyset \text{ whenever } z_1, z_2 \in \mathbb{Z}^N,\ z_1 \neq z_2 \tag{5.2}$$

(P_{E^+} is the orthogonal projection on E^+; these conditions correspond to (9) and (10) in [AS]). Clearly, J is even, $\mathcal{K} = -\mathcal{K}$ and $\mathcal{F} = -\mathcal{F}$ (in the language of [AS], J is invariant with respect to the representation R of $\mathbb{Z}_2 = \{-1, 1\}$ given by $R_{-1}u = -u$; moreover, $E^{\mathbb{Z}_2} = \{0\}$ and $R \subset S$). *From now on we consider J as an even functional and disregard the S^1-invariance.* Let

$$\mathcal{U}_\delta := E^- \oplus \bigcup_{z\in\mathbb{Z}^N} \{u^+ \in E^+ : d(u^+, T_z\mathcal{F}) < \delta\},$$

where $d(u, A)$ denotes the distance from u to the set A and let $\mathcal{H}$ be the class of mappings $f : E \to E$ such that f is a homeomorphism, $f(-u) = -f(u)$ for all u and $f(J^c) \subset J^c$ for all $c \ge -1$ (as usual, $J^c := \{u \in E : J(u) \le c\}$). Taking a smaller ρ in (4.3) if necessary we may assume that

$$\inf_{u\in K_J\setminus\{0\}} J(u) > d. \tag{5.3}$$

Lemma 5.1. *Suppose J satisfies (5.1), (5.2) and $c \ge d$, where d is as in (4.3), and (5.3) is satisfied. For each $\delta > 0$ small enough there exists $\varepsilon_0 > 0$ such that whenever $0 < \varepsilon < \varepsilon_0$, then there is a mapping $f \in \mathcal{H}$ with $f(J^{c+\varepsilon} \setminus \mathcal{U}_\delta) \subset J^{c-\varepsilon}$.*

This is a variant of the deformation lemma which will be needed in the minimax procedure below. The argument is the same as in Lemma 5.3 in [AS]. The proof requires two auxiliary results corresponding to Lemmas 5.1 and 5.2 there (the translation by the elements of $\mathbb{Z}$ in [AS] should be replaced by the operators T_z, $z \in \mathbb{Z}^N$). An important role in obtaining a result which corresponds to Lemma 5.1 in [AS] is played by the fact that whenever $|z_n| \to \infty$, then $T_{z_n}u \rightharpoonup 0$ for each fixed u, and by the following lemma describing the behavior of the Palais-Smale sequences:

Lemma 5.2. *Suppose J satisfies the hypotheses of Theorem 1.4 and let $\{u_n\}$ be a $(PS)_c$-sequence. Then, up to a subsequence, either $u_n \to 0$ (and $c = 0$) or $c \ge d$*

and there exist $\bar u_1, \dots, \bar u_l \in K_J \setminus \{0\}$ *and sequences* $\{z_n^j\} \subset \mathbb{Z}^N$ $(1 \le j \le l)$ *such that*

$$\left\|u_n - \sum_{j=1}^{l} T_{z_n^j}\bar u_j\right\| \to 0, \quad ||z_n^j - z_n^k|| \to \infty \text{ as } n \to \infty \text{ and } j \neq k$$

and

$$\sum_{j=1}^{l} J(\bar u_j) = c.$$

This lemma is an adaptation to our case of a well-known result which may be found e.g. in [CR, KS] (see also [AS]). The proof follows that of Proposition 4.2 in [KS] with one exception. We do not know whether $v(x) \to 0$ as $|x| \to \infty$ if $v \in K_J \setminus \{0\}$. However, $v \in L^q(\mathbb{R}^N)$ for any $q \ge 2$. Indeed, we have

$$-\Delta_A v + v = (1 - V(x))v + g(x, |v|)v \in L^{2^*}(\mathbb{R}^N) + L^{2^*/(p-1)}(\mathbb{R}^N) \tag{5.4}$$

(we assume $N \ge 3$, the case $N = 2$ being simpler). Hence, according to Corollary $B.13.3$ in [S], $v \in L^{\tilde q_1}$ whenever

$$\frac{1}{2^*} - \frac{1}{\tilde q_1} < \frac{2}{N} \quad \text{and} \quad \frac{1}{q_0} - \frac{1}{\tilde q_1} < \frac{2}{N}$$

(here $q_0 := \frac{2^*}{p-1}$). Since $2 < p < 2^*$, it is easy to see that we can choose $\tilde q_1 > 2^*$. Then the right-hand side of (5.4) is in $L^{\tilde q_1}(\mathbb{R}^N) + L^{q_1}(\mathbb{R}^N)$, where $q_1 := \frac{\tilde q_1}{p-1}$. Bootstraping we obtain $v \in L^q(\mathbb{R}^N)$ for any $q > 2^*$, and hence for any $q \ge 2$.

Now if $v \in K_J \setminus \{0\}$, then for each $q \in (2, +\infty)$ and $\varepsilon > 0$ we can find a bounded domain Ω such that $||v||_{H^1_A(\mathbb{R}^N\setminus\Omega)} \le \varepsilon$, $\int_{\mathbb{R}^N\setminus\Omega} F(x, |v|) \le \varepsilon$, $||v||_{L^2(\mathbb{R}^N\setminus\Omega)} \le \varepsilon$ and $||v||_{L^q(\mathbb{R}^N\setminus\Omega)} \le \varepsilon$. Since $q < +\infty$, we need to modify the estimate (4.20) in [KS]. More precisely, we must show that

$$\int_{\mathbb{R}^N\setminus\Omega} \big|g(x, |w+v|)(w+v) - g(x, |w|)w\big|\,|\varphi| \le c_0\varepsilon, \tag{5.5}$$

where c_0 is independent of w and φ as long as $\|w\|$ is uniformly bounded by some constant $\tilde c$ and $\|\varphi\| \le 1$. Let $\omega := \{x \in \mathbb{R}^N : |v(x)| > \varepsilon_0\}$, where ε_0 is taken from (A6). Since $v \in L^q(\mathbb{R}^N)$, $\mu(\omega) < \infty$. We may assume $\Omega = B(0, R)$ and it is clear that $\mu(\omega \setminus \Omega) \to 0$ as $R \to \infty$. Using (A6), the Hölder, the Sobolev and the diamagnetic inequalities and choosing q such that $\frac{p}{2^*} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus(\Omega\cup\omega)} \big|g(x, |w+v|)(w+v) - g(x, |w|)w\big|\,|\varphi| &\le \bar C \int_{\mathbb{R}^N\setminus(\Omega\cup\omega)} (1 + |w|^{p-1})|v|\,|\varphi| \\ &\le \bar C(\|v\|_{L^2(\mathbb{R}^N\setminus\Omega)}\|\varphi\|_2 + \|w\|_{2^*}^{p-1}\|v\|_{L^q(\mathbb{R}^N\setminus\Omega)}\|\varphi\|_{2^*}) \le c_1\varepsilon.\end{aligned}$$

Moreover, by (A4) there exists C_1 such that if $\omega_1 := \omega \setminus \Omega$, then

$$\begin{aligned}\int_{\omega_1} \big|g(x, |w+v|)(w+v) - g(x, |w|)w\big|\,|\varphi| &\le C_1 \int_{\omega_1} (|v| + |w| + |v|^{p-1} + |w|^{p-1})|\varphi| \\ &\le C_1\mu(\omega_1)^{2/N}(\|v\|_{2^*} + \|w\|_{2^*})\|\varphi\|_{2^*} + C_1\mu(\omega_1)^{1/q}(\|v\|_{2^*}^{p-1} + \|w\|_{2^*}^{p-1})\|\varphi\|_{2^*}.\end{aligned}$$

Since $\mu(\omega_1) \to 0$ as $R \to \infty$, R may be chosen so that the right-hand side above is less than ε. Hence (5.5) holds with $c_0 = c_1 + 1$.

Having our Lemma 5.2, the arguments of Lemmas 5.1 and 5.2 in [AS] go through unchanged as does the argument of Lemma 5.3 there. This concludes the brief summary of the proof of Lemma 5.1.

Let $\rho > 0$ in (4.3) be chosen in such a way that (5.3) holds and $J|_{\bar{B}(0,\rho)} > -1$. For A closed and symmetric (i.e. $A = -A$) we define

$$\gamma^*(A) = \min_{f\in\mathcal{H}} \gamma(f(A) \cap \partial B(0,\rho) \cap E^+),$$

where γ is Krasnoselskii's genus (γ^* is a variant of Benci's pseudoindex [B]). Let

$$d_k := \inf_{\gamma^*(A)\geq k} \sup_{u\in A} J(u).$$

Since $J|_{\partial B(0,\rho)\cap E^+} \geq d$, $d_k \geq d$. Moreover, for each $\delta > 0$ we have

$$J(u) \leq \frac{1}{2}\left|\left|u^+\right|\right|^2 - \frac{1}{2}\left|\left|u^-\right|\right|^2 - c(\delta)\left|\left|u\right|\right|^\mu_{L^\mu(\mathbb{R}^N)} + \delta\left|\left|u\right|\right|^2_{L^2(\mathbb{R}^N)}$$

(cf. (1.7) in [KS]), hence $J(u) \to -\infty$ as $||u|| \to \infty$, $u \in E_k^+ \oplus E^-$, where $E_k^+ \subset E^+$ and $\dim E_k^+ = k$. Therefore there exist sets of arbitrarily large pseudoindex (cf. [AS], Lemma 4.6 or [KS], Lemma 4.8) and d_k is defined for all $k \geq 1$. Since $d_0 := \sup_{\mathcal{U}_\delta} J < \infty$, it follows from Lemma 5.1 that $d_k \leq d_0$ for all k; consequently, $d_k \nearrow \bar{d} \leq d_0$. As $\gamma(S^1) = 2$, it is easy to see that $\gamma(\mathcal{F}) = 2$ and $\gamma(\bar{\mathcal{U}}_\delta) = 2$ provided δ is small enough. Using Lemma 5.1 once more we obtain

$$k \leq \gamma^*(J^{d_k+\varepsilon}) \leq \gamma^*(J^{d_k+\varepsilon} \setminus \mathcal{U}_\delta) + \gamma(\bar{\mathcal{U}}_\delta) \leq \gamma^*(J^{d_k-\varepsilon}) + 2.$$

Therefore $\gamma^*(J^{d_k-\varepsilon}) \geq k-2$, so $d_k - \varepsilon \geq d_{k-2}$ and $\bar{d} - \varepsilon \geq \bar{d}$, a contradiction. Hence there is no compact set $\mathcal{K}$ satisfying (5.1) and (5.2). This completes the proof. More details may be found in the proof of Theorem 6.1 in [AS].

Appendix

The proof of Lemma 3.1 in the case $A \equiv 0$ can be found in Willem's book [W]. In the following we highlight the main points in the proof with nontrivial magnetic potential.

Assume first that $u = 0$, then for all $h \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$,

$$\int_{\mathbb{R}^N} |\nabla_A(hu_n)|^2 = \int_{\mathbb{R}^N} |h\nabla_A u_n + u_n \nabla h|^2 =$$

$$\int_{\mathbb{R}^N} |h\nabla_A u_n|^2 + |u_n\nabla h|^2 + 2\operatorname{Re}\int_{\mathbb{R}^N} h\bar{u}_n \nabla_A u_n \cdot \nabla h \to \int_{\mathbb{R}^N} |h|^2 \, d\mu$$

because $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$ according to Lemma 2.6. Moreover, by the Sobolev and the diamagnetic inequalities,

$$\left(\int_{\mathbb{R}^N} |hu_n|^{2^*}\right)^{2/2^*} \leq S^{-1}\int_{\mathbb{R}^N} |\nabla_A(hu_n)|^2,$$

therefore

$$\left(\int_{\mathbb{R}^N} |h|^{2^*} \, d\nu\right)^{2/2^*} \leq S^{-1}\left(\int_{\mathbb{R}^N} |h|^2 \, d\mu\right)$$

and (1) of Lemma 3.1 follows.

Let $\psi_R \in C^\infty(\mathbb{R}^N, [0,1])$ be such that $\psi_R(x) = 1$ if $|x| \geq R+1$ and $\psi_R(x) = 0$ if $|x| \leq R$. Then

$$\left(\int_{\mathbb{R}^N} |\psi_R u_n|^{2^*}\right)^{2/2^*} \leq S^{-1} \int_{\mathbb{R}^N} |\nabla_A (\psi_R u_n)|^2$$

and since $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$, $\nabla \psi_R$ has compact support and

$$\nabla_A(\psi_R u_n) = \psi_R \nabla_A u_n + u_n \nabla \psi_R,$$

we have

$$\limsup_{n\to\infty} \left(\int_{\mathbb{R}^N} |\psi_R u_n|^{2^*}\right)^{2/2^*} \leq S^{-1} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla_A u_n|^2 \psi_R^2. \tag{A.1}$$

Following [W] we obtain

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla_A u_n|^2 \psi_R^2, \quad \nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \psi_R^{2^*}$$

and (2) of Lemma 3.1 follows from (A.1).

The proof that if $||\nu||^{2/2^*} = S^{-1} ||\mu||$, then μ and ν are concentrated at a single point is exactly the same as in [W].

We consider now the case when $u \neq 0$. Let $v_n := u_n - u$. Then $v_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, so (1) is satisfied also if $u \neq 0$ because, as we already have shown, the corresponding inequality holds for $\{v_n\}$. Furthermore,

$$\limsup_{n\to\infty} \int_{|x|\geq R} |\nabla_A v_n|^2 = \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla_A u_n|^2 - \int_{|x|\geq R} |\nabla_A u|^2 ,$$

therefore

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla_A v_n|^2 ,$$

and using the Brézis-Lieb lemma as in [W],

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |v_n|^{2^*} .$$

So also (2) follows from the corresponding inequality for $\{v_n\}$.

Let ψ_R be as above and set $h = 1 - \psi_R$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_A u_n|^2 h &= \int_{\mathbb{R}^N} |\nabla_A (v_n + u)|^2 h \\ &= \int_{\mathbb{R}^N} |\nabla_A v_n|^2 h + |\nabla_A u|^2 h + 2\mathrm{Re} \int_{\mathbb{R}^N} h \nabla_A v_n \cdot \overline{\nabla_A u} \\ &\to \int_{\mathbb{R}^N} h\, d\mu + \int_{\mathbb{R}^N} |\nabla_A u|^2 h.\end{aligned}$$

Using the Brézis-Lieb lemma as in [W] again we also have

$$\int_{\mathbb{R}^N} |u_n|^{2^*} h \to \int_{\mathbb{R}^N} h\, d\nu + \int_{\mathbb{R}^N} |u|^{2^*} h.$$

It follows that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla_A u_n|^2 = \limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}\psi_R\,|\nabla_A u_n|^2 + \int_{\mathbb{R}^N}(1-\psi_R)\,|\nabla_A u_n|^2\right)$$
$$= \limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}\psi_R\,|\nabla_A u_n|^2\right) + \int_{\mathbb{R}^N}(1-\psi_R)d\mu + \int_{\mathbb{R}^N}(1-\psi_R)\,|\nabla_A u|^2$$

and when $R\to\infty$ we get, by Lebesgue's dominated convergence theorem,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla_A u_n|^2 = \mu_\infty + \int_{\mathbb{R}^N}d\mu + \int_{\mathbb{R}^N}|\nabla_A u|^2 = \mu_\infty + ||\mu|| + ||\nabla_A u||_2^2\,.$$

This proves (3). The proof of (4) is similar.

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