Heuristics for Swedish style bond futures

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Abstract

Swedish bond futures have unique features that we analyse. We show by no-arbitrage arguments that the contract works as intended and provide theoretical justification for the heuristics already used in the market with regard to hedge ratio, and the relation between the quoted yield on a futures contract and the forward yield of the underlying bond.

1 Introduction

Swedish bond futures contracts have some peculiar features that are not found in any other large bond market. They are quoted on yield rather than price, just as Australian bond futures. The daily settlement is based on a synthetic underlying as most futures, American treasury futures being the most prominent example. The contract has physical delivery of the underlying bond as is common in most other markets (e.g. the U.S.), though the other main yield quoted futures contracts, for Australian bonds, are cash settled. However, there is only one single eligible bond per contract, rather than a basket. The combination of quotation on yield, synthetic underlying and a single physical deliverable bond, makes the contract unique and analyses for other futures contracts without one or more of these features are not necessarily applicable to this contract.

In particular, the combination of a synthetic underlying with physical delivery of a single bond raises some questions about the dynamics of the futures contract. We know that the contract specification works in practice since the open interest is usually of the order 100 billion SEK so that any substantial defects in the specifications would already have been found by market participants.

Nevertheless, we show that the contract design ensures that the yield of the futures contract converges to that of the underlying bond at delivery, and that the yield before the time of delivery is equal to the theoretical forward yield of the underlying bond plus a possible correction term that in usual circumstances is less than a basis point in magnitude. As a by-product of this analysis, we also affirm that the hedge ratio between futures contract and underlying bond used in the market is correct up to a linear approximation, and understand that the carry from a position in futures is approximately equal to that of a forward position in the underlying bond, scaled with the hedge ratio.

We start by describing the futures contract in more detail and then show that the yield of the futures contract should converge to that of the deliverable bond at the final fix. We note that this is a somewhat hypothetical issue since no contract since 2015 — when the the contracts with daily rather than monthly settlement were introduced — has been held until expiry; all contracts have been closed or rolled into the contract with the next expiry. Then we present the well-known approximation of bond prices with duration and convexity. For our purposes this second degree approximation is shown to be highly accurate. Market practice is to use only a duration approximation, i.e. a linear, first order approximation, when using these instruments, say for calculating hedge ratios. Assuming this approximation to be good enough, we show by an arbitrage argument that the yield of the futures contract is essentially equal to the forward yield of the deliverable bond. Next we add the convexity term and show that this convexity adjustment to the previous arbitrage argument is practically negligible.

2 Description of the futures contract

The Swedish government bond market uses annually compounded interest rates and coupons are paid annually. The invoice price of a bond with face value 1, coupon a, yield y, and time to maturity M measured in years is

$$P(y; a, M) = (1+y)^{\lceil M \rceil - M} \frac{a}{y} \left(1 - \frac{1}{(1+y)^{\lceil M \rceil}}\right) + \frac{1}{(1+y)^M}. \quad (1)$$

The futures contract works as follows, see [4, chapter 3.72] for the exact definitions.

The underlying deliverable bond is determined as the government bond with remaining time to maturity closest to 2, 5, and 10 years respectively, at the time of expiry. There are also futures contracts with similar design for covered bonds with 2 and 5 years maturity from the largest Swedish banks.

Each contract is for the face amount of 1 million SEK of the underlying. The final fix is determined as the median of the market makers' mid quotes in yield. If you hold a long (short) position in the contract at the time of expiry you will buy (sell) the underlying bond for an invoice price determined by the formula (1). As inputs to the formula, y is the aforementioned fixing yield, a is the coupon rate of the underlying bond and M its time to maturity.

However, the daily settlement of the contract works as follows: If you enter the contract, say in a long position, at yield y_0 and the daily fix at market close is y_1 (determined in the same way as the final fix), then you will receive a daily cash settlement determined as $P(y_1; 0.06, M') - P(y_0, 0.06, M')$ (interpret as paying if negative), where M' is 2, 5, or 10. This is also the cash settlement if you hold the long position over a day and y_0 is yesterday's daily fix. We refer to this daily payment as the variation margin.

Note that the coupon rate is always 6% regardless of the coupon rate of the deliverable bond and that M' also typically differs from the time to maturity of the deliverable bond. That M' is fixed during the life-time of the futures contract means P can be interpreted as a forward price, i.e. a price for buying a bond with forward settlement (at the time of expiry) that will have M' as it remaining maturity at that time. This contract base is called a synthetic bond in [4].

If you are long a single contract until expiry, you will end up owning the deliverable bond with a face amount of 1 million SEK. However, the profit or loss associated with this trade is not the same as the one you would get by buying the bond outright, or even by buying it with forward settlement, and this is due to the variation margin being based on a synthetic bond.

One may compare this to American treasury futures. These trade on price of a synthetic bond rather than yield. The daily settlement is based on this synthetic bond. The futures price is adjusted with bond specific conversion factors to arrive at the different invoice prices for the deliverable bonds. A short position in a treasury future entails some degree of optionality since there is a choice of what bond from the basket of eligible bonds to deliver and also a choice of time when to deliver. Swedish bond futures have no such optionality.

Swedish bond futures have some similarities with Australian bond futures in that they are quoted on yield and that the contract base is synthetic, and it differs in that the Australian contracts are cash settled and the final fix is based on the average yield of the underlying basket.

3 Convergence at the final fix

A simple argument shows that the yield of the futures contract must converge to that of the deliverable bond at the time of expiry.

The fix at the end of each trading session is determined as the average of mid quotes of the market makers. If one buys (sells) the contract exactly when the final fix is determined, the yield that one trades on is at most half the bid-offer spread from the final fix. This means that the variation margin, which depends on the synthetic, is minimal, and that one will buy (sell) the underlying bond for a price corresponding to the yield determined in the final fix.

Since one could conceivably buy (sell) the bond for its true yield at the same time, there would be an arbitrage if the yield of the futures contract would not be within the bid-offer of the underlying.

4 Approximating the bond price

The price of a bond is negatively and convexly related to its yield.

With $P(y)$ being the price of a bond as a function of its yield

$$P(y+dy) = P(y)\left(1 - Ddy + \tfrac{1}{2}c(dy)^2 + O(|dy|^3)\right),$$

where D is the bond's duration and c its convexity. Both D and c depend on y, even though we have suppressed that in the notation.

We rather use dollar duration (or risk) $R = D \cdot P(y)$ and dollar convexity $C = c \cdot P(y)$ of a bond with face value one:

$$P(y+dy) - P(y) = -Rdy + \tfrac{1}{2}C(dy)^2 + O(|dy|^3). \qquad (2)$$

Closed formulas for R and D can be obtained by differentiating (1) analytically, though in practice numerical differentiation of (1) works just as fine.

The approximation is generally worse for larger times to maturity. Let us consider time to maturity 10 years, yields and coupons in the interval $[0\%, 12\%]$, and yield shifts $|dy| \leq 1\%$, and compare (1) with the second order approximation (2) on a fine grid of yields, coupons and yield shifts. The largest approximation error in price is then 0.02%, and the largest approximation error in yield is 0.002 percentage points, i.e. 0.2 basis points. For our purposes, and for most market participants, this is negligible.

We are mostly interested in time horizons up to three months since the futures contracts expire every three months and essentially all trading is concentrated in the contract with closest time to expiry, except for the last week before expiry when market participants roll into the next contract. Over this time horizon it is very unlikely to see a larger yield shift $|dy|$ than 1%. Even for yield shifts $|dy| \leq 2\%$, the largest approximation error in price is 0.2% and the approximation error in yield is 2 basis points.

5 Assuming the linear approximation to be exact

The convexity is often small and assuming the bond price is linear in the yield, we have that the yield of the contract is identical to the forward yield of the underlying for all times before the fix. This can be proved by the following no arbitrage argument.

Let R^s be the dollar duration of the synthetic bond and R the dollar duration of the underlying, and similarly y_t^s and y_t the respective forward yields at time $t \leq T$ for delivery at time T, the settlement date. Also let $P^s(y)$ and $P(y)$ be the price at time T as functions of their respective yields.

The linearity of the price means that

$$P(y_T) = P(y_0) - R(y_T - y_0)$$
$$P^s(y_T^s) = P^s(y_0^s) - R^s(y_T^s - y_0^s).$$

Linearity also means that R and R^s are constants not depending on y_0 and y_0^s.

We consider $t < T$. We will assume $y_t^s > y_t$ and derive an arbitrage. In the case $y_t^s < y_t$, exchange all references to selling below with buying and vice versa.

1. At time t, buy R contracts and sell R^s underlying bonds with delivery T.
2. At time T, buy $R^s - R$ additional contracts (interpret as selling if $R^s - R$ is negative) so that the final position is long R^s contracts and short R^s bonds.

We can deliver the bonds we receive through the contracts to our counterparty in the bond sale. Step 1. means that the *hedge ratio* is $h := R^s/R$, i.e. you need h futures contracts to have the same interest rate exposure as 1 million SEK of face value of the deliverable bond. The hedge ratio is used like this in practice, but with R^s and R calculated at the current respective yields. Also note that hedging treasury bonds with treasury futures produces a similar *delivery tail* as in step 2. above, see [2].

Our profit and loss will consist of three parts, firstly the variation margin of the futures contracts, secondly the invoice we have to pay for the bonds received through the contract, thirdly we receive payment for the bonds we sold at time t (and delivered at T).

1. The variation margin accumulated to time T of a single contract bought at time t is $-R^s(y_T - y_t^s)$ (recall $y_T^s = y_T$) and the total for our R contracts that we bought at time t is thus $-RR^s(y_T - y_t^s)$. The variation margin is zero for the $R^s - R$ contracts entered at the final fix.
2. We pay $P(y_T) = P(y_t) - R(y_T - y_t)$ for each of the R^s bonds we receive through the contracts for a total of $R^s P(y_t) - RR^s(y_T - y_t)$.
3. We receive a payment of $P(y_t)$ for each of the R^s bonds we sold at time t (and deliver at time T), for a total of $R^s P(y_t)$.

Net, we get

$$-RR^s(y_T - y_t^s) - (R^s P(y_t) - RR^s(y_T - y_t)) + R^s P(y_t) = RR^s(y_t^s - y_t) > 0,$$

i.e. an arbitrage. Therefore, the market is arbitrage free only if $y_t^s = y_t$ for all t.

Remark 1. One might argue that we have not accounted for the daily cash flows due to the futures contracts' variation margin. However, the forward sale of the R^s bonds at time t would realistically be done with with a bilateral agreement to post collateral until the settlement at time T, and the daily flow of collateral would more or less net the daily flow of variation margin.

Remark 2. Market practice is to use the hedge ratio as defined above, though to calculate it at the prevailing forward yield for the futures contract and the spot yield of the underlying bond. As the spot yield is typically close to the forward yield (recall that the time to expiry is relatively short), and the quadratic approximation does not improve much on the linear (see the next section), we realise that the market practice is well supported by the theoretical arguments.

6 Assuming the quadratic approximation to be exact

Now let us assume that the forward bond price is quadratic in the yield so that

$$\begin{aligned} P(y_T) &= P(y_0) - R(y_T - y_0) + \tfrac{1}{2}C(y_T - y_0)^2 \\ P^s(y_T) &= P^s(y_0) - R^s(y_T - y_0) + \tfrac{1}{2}C^s(y_T - y_0)^2 \end{aligned}$$

for a fixed y_0. We know from Section 4 that the approximation is very good when $|y_T - y_0| \leq 1\%$.

Since $P(y_0)$ is the price at time 0 for delivery at time T we have $P(y_0) = \mathbb{E}^{\mathbb{Q}}[P(y_T)]$, where $\mathbb{E}^{\mathbb{Q}}$ is the expected value under the T-forward measure $\mathbb{Q}$. If we take the realistic approach to collateral raised in Remark 1, the forward purchase will produce a collateral flow as from a futures contract, which means that $\mathbb{Q}$ instead is the usual risk neutral measure rather than the T-forward measure, see [1, Ch. 20]. The difference is not material for our purposes.

We have

$$P(y_0) = \mathbb{E}^{\mathbb{Q}}[P(y_T)] = P(y_0) - R\mathbb{E}^{\mathbb{Q}}[y_T - y_0] + \tfrac{1}{2}C \underbrace{\mathbb{E}^{\mathbb{Q}}[(y_T - y_0)^2]}_{=:\tau^2}$$

$$\Rightarrow$$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[y_T - y_0] &= \frac{1}{2}\frac{C}{R}\tau^2 \\ \mathbb{E}^{\mathbb{Q}}[y_T] &= y_0 + \frac{1}{2}\frac{C}{R}\tau^2. \end{aligned}$$

Due to the quadratic term when $C \neq 0$, the expected value of the rate y_T is not identical to y_0 as it is when assuming that the bond price is linear in the yield.

We do similar calculations for the synthetic bond.

$$\begin{aligned} P^s(y_0^s) &= \mathbb{E}^{\mathbb{Q}}[P^s(y_T^s)] = \mathbb{E}^{\mathbb{Q}}[P^s(y_T)] \\ &= \mathbb{E}^{\mathbb{Q}}[P^s(y_0) - R^s(y_T - y_0) + \tfrac{1}{2}C^s(y_T - y_0)^2] \\ &= P^s(y_0) - \frac{1}{2}R^s\frac{C}{R}\tau^2 + \frac{1}{2}C^s\tau^2 \\ &= P^s(y_0) - \frac{1}{2}\left(\frac{R^s}{R}C - C^s\right)\tau^2 \end{aligned} \tag{3}$$

Note that $P^s(y)$ is decreasing in y, and since τ^2 and C^s are positive,

$$y_0^s > y_0 \iff \frac{R^s}{R}C > C^s \iff \frac{R^s/C^s}{R/C} > 1. \tag{4}$$

We also have

$$P^s(y_0^s) = P^s(y_0) - R^s(y_0^s - y_0) + \frac{1}{2}C^s(y_T^s - y_0)^2.$$

Combining this with (3) we get

$$\frac{1}{2}C^s(y_0^s - y_0)^2 - R^s(y_0^s - y_0) + \frac{1}{2}\left(\frac{R^s}{R}C - C^s\right)\tau^2 = 0.$$

This second degree equation has the solutions

$$\begin{aligned} y_0^s - y_0 &= \frac{R^s}{C^s} \pm \sqrt{\left(\frac{R^s}{C^s}\right)^2 - \left(\frac{R^s/C^s}{R/C} - 1\right)\tau^2} \\ &= \frac{R^s}{C^s}\left(1 \pm \sqrt{1 - \frac{C^s}{R^s}\left(\frac{C}{R} - \frac{C^s}{R^s}\right)\tau^2}\right). \end{aligned}$$

However, we seek only the negative root due to (4). Hence

$$\begin{aligned} y_0^s &= y_0 + \frac{R^s}{C^s}\left(1 - \sqrt{1 - \frac{C^s}{R^s}\left(\frac{C}{R} - \frac{C^s}{R^s}\right)\tau^2}\right) \\ &\approx y_0 + \frac{1}{2}\left(\frac{C}{R} - \frac{C^s}{R^s}\right)\tau^2 \end{aligned}$$

Note that $\tau^2 \approx \sigma^2 := \mathbb{E}^{\mathbb{Q}}[(y_T - \mathbb{E}^{\mathbb{Q}}[y_T])^2]$, since

$$\begin{aligned} \sigma^2 &= \mathbb{E}^{\mathbb{Q}}[(y_T - \mathbb{E}^{\mathbb{Q}}[y_T])^2] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\left(y_T - y_0 - \frac{1}{2}\frac{C}{R}\tau^2\right)^2\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[(y_T - y_0)^2 - \frac{C}{R}\tau^2(y_T - y_0) + \frac{1}{4}\left(\frac{C}{R}\right)^2\tau^4\right] \\ &= \tau^2 - \frac{1}{4}\left(\frac{C}{R}\right)^2\tau^4. \end{aligned}$$

There exist options with Swedish bond futures as underlying but they are hardly ever traded in significant size. If we use swaptions as a proxy we may estimate σ to be at most of the order 1% on a yearly basis, and thus on the order $1\%/\sqrt{4} = 0.5\%$ on a quarterly basis. σ^2 and τ^2 are both thus of the order a quarter of a basis point. The debt profile of Swedish government bonds is such that bonds mature roughly every 18 months out to 10 years into the future and less frequently further out. This means that the deliverable bond differs at most circa 9 months in time to maturity from the synthetic underlying the futures contract, and always less than one. If the underlying differs with at most one year in time to maturity from the synthetic bond, and if yields and coupons are in the interval $[0\%, 12\%]$, then $|\frac{C}{R} - \frac{C^s}{R^s}| < 2$. Therefore the "convexity correction" term $\frac{1}{2}(\frac{C}{R} - \frac{C^s}{R^s})\tau^2$ is at most a quarter of a basis point in normal situations, which is practically negligible.

7 Carry

Following [3], we define an asset's carry as its futures return, assuming the price stays the same. For bonds this means that the prices of bonds with the same time to maturity — not time of maturity — are constant as time passes. We know from Section 3 that the yield of the futures contract converges to that of the underlying deliverable bond at time of expiry.

We have argued that the yield of the Swedish bond futures is approximately equal to the forward yield of the deliverable government bond, y. Let z_m be the prevailing spot yield for time to maturity m, so that the deliverable bond with, say, time to maturity M, has spot yield y_M. Assuming a constant yield curve, it will have spot yield $y_{M-\tau}$ in τ years when the futures contract expires, and the profit from being long a futures contract would be approximately

$$R^s(y - z_{M-\tau}) = R^s(y - z_M) + R^s(z_M - z_{M-\tau})$$

using the duration approximation. In practice only y and z_M are directly observable in the market and the slope $z_M - z_{M-\tau}$, often positive, must be estimated by interpolation. Note that

$$R^s(y - z_{M-\tau}) = hR(y - z_{M-\tau}),$$

i.e. the carry of the futures position is approximately equal to the carry of a forward position in the underlying bond, scaled with the hedge ratio.

8 Conclusions

We have presented the first theoretical study of Swedish style bond futures that are quoted on yield, have daily settlement against a synthetic underlying, and physical delivery of a single eligible bond. We have shown, not surprisingly, that their yield converges to that of the underlying and that the hedge ratio as calculated by market participants is essentially correct. Our other contributions such as the observation that the yield of the futures contract equals that of the forward yield of the deliverable bond plus a convexity correction term that is negligible in most circumstances, and the expression for the approximate carry of a futures position, might be novel even for some users of the instruments.

References

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